

HANDBOOK OF

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**LINEAR PARTIAL  
DIFFERENTIAL  
EQUATIONS for  
ENGINEERS  
and SCIENTISTS**

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Andrei D. Polyanin



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# FOREWORD

Linear partial differential equations arise in various fields of science and numerous applications, e.g., heat and mass transfer theory, wave theory, hydrodynamics, aerodynamics, elasticity, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, control theory, chemical engineering sciences, and biomechanics.

This book presents brief statements and exact solutions of more than 2000 linear equations and problems of mathematical physics. Nonstationary and stationary equations with constant and variable coefficients of parabolic, hyperbolic, and elliptic types are considered. A number of new solutions to linear equations and boundary value problems are described. Special attention is paid to equations and problems of general form that depend on arbitrary functions. Formulas for the effective construction of solutions to nonhomogeneous boundary value problems of various types are given. We consider second-order and higher-order equations as well as the corresponding boundary value problems. All in all, the handbook presents more equations and problems of mathematical physics than any other book currently available.

For the reader's convenience, the introduction outlines some definitions and basic equations, problems, and methods of mathematical physics. It also gives useful formulas that enable one to express solutions to stationary and nonstationary boundary value problems of general form in terms of the Green's function.

Two supplements are given at the end of the book. Supplement A lists properties of the most common special functions (the gamma function, Bessel functions, degenerate hypergeometric functions, Mathieu functions, etc.). Supplement B describes the methods of generalized and functional separation of variables for nonlinear partial differential equations. We give specific examples and an overview application of these methods to construct exact solutions for various classes of second-, third-, fourth-, and higher-order equations (in total, about 150 nonlinear equations with solutions are described). Special attention is paid to equations of heat and mass transfer theory, wave theory, and hydrodynamics as well as to mathematical physics equations of general form that involve arbitrary functions.

The equations in all chapters are in ascending order of complexity. Many sections can be read independently, which facilitates working with the material. An extended table of contents will help the reader find the desired equations and boundary value problems. We refer to specific equations using notation like "1.8.5.2," which means "Equation 2 in Subsection 1.8.5."

To extend the range of potential readers with diverse mathematical backgrounds, the author strove to avoid the use of special terminology wherever possible. For this reason, some results are presented schematically, in a simplified manner (without details), which is however quite sufficient in most applications.

Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics.

The author thanks Alexei Zhurov for useful remarks on the manuscript.

The author hopes that the handbook will be useful for a wide range of scientists, university teachers, engineers, and students in various areas of mathematics, physics, mechanics, control, and engineering sciences.

*Andrei D. Polyinin*

# BASIC NOTATION

## Latin Characters

$\mathcal{E}$	fundamental solution
$\text{Im}[A]$	imaginary part of a complex quantity $A$
$G$	Green's function
$\mathbb{R}^n$	$n$ -dimensional Euclidean space, $\mathbb{R}^n = \{-\infty < x_k < \infty; k = 1, \dots, n\}$
$\text{Re}[A]$	real part of a complex quantity $A$
$r, \varphi, z$	cylindrical coordinates, $r = \sqrt{x^2 + y^2}$ and $x = r \cos \varphi, y = r \sin \varphi$
$r, \theta, \varphi$	spherical coordinates, $r = \sqrt{x^2 + y^2 + z^2}$ and $x = r \sin \theta \cos \varphi, y = r \sin \theta \sin \varphi, z = r \cos \theta$
$t$	time ( $t \geq 0$ )
$w$	unknown function (dependent variable)
$x, y, z$	space (Cartesian) coordinates
$x_1, \dots, x_n$	Cartesian coordinates in $n$ -dimensional space
$\mathbf{x}$	$n$ -dimensional vector, $\mathbf{x} = \{x_1, \dots, x_n\}$
$ \mathbf{x} $	magnitude (length) of $n$ -dimensional vector, $ \mathbf{x}  = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$
$\mathbf{y}$	$n$ -dimensional vector, $\mathbf{y} = \{y_1, \dots, y_n\}$

## Greek Characters

$\Delta$	Laplace operator
$\Delta_2$	two-dimensional Laplace operator, $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
$\Delta_3$	three-dimensional Laplace operator, $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
$\Delta_n$	$n$ -dimensional Laplace operator, $\Delta_n = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$
$\delta(x)$	Dirac delta function; $\int_{-a}^a f(y)\delta(x-y) dy = f(x)$ , where $f(x)$ is any continuous function, $a > 0$
$\delta_{nm}$	Kronecker delta, $\delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}$
$\vartheta(x)$	Heaviside unit step function, $\vartheta(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$

## Brief Notation for Derivatives

$$\partial_t w = \frac{\partial w}{\partial t}, \quad \partial_x w = \frac{\partial w}{\partial x}, \quad \partial_{tt} w = \frac{\partial^2 w}{\partial t^2}, \quad \partial_{xx} w = \frac{\partial^2 w}{\partial x^2} \quad (\text{partial derivatives})$$

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f_x^{(n)} = \frac{d^n f}{dx^n} \quad (\text{derivatives for } f = f(x))$$

## Special Functions (See Also Supplement A)

$$\text{Ai}(x) = \frac{1}{\pi} \int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$$

$$\text{Airy function; } \text{Ai}(x) = \frac{1}{\pi} \sqrt{\frac{1}{3}x} K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$$

$$\text{Ce}_{2n+p}(x, q) = \sum_{k=0}^{\infty} A_{2k+p}^{2n+p} \cosh[(2k+p)x]$$

$$\text{even modified Mathieu functions, where } p = 0, 1;$$

$$\text{Ce}_{2n+p}(x, q) = \text{ce}_{2n+p}(ix, q)$$

$$ce_{2n}(x, q) = \sum_{k=0}^{\infty} A_{2k}^{2n} \cos 2kx$$

$$ce_{2n+1}(x, q) = \sum_{k=0}^{\infty} A_{2k+1}^{2n+1} \cos[(2k+1)x]$$

$$D_\nu = D_\nu(x)$$

$$\operatorname{erf} x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-\xi^2) d\xi$$

$$\operatorname{erfc} x = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\xi^2) d\xi$$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$$

$$F(a, b, c; x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

$$I_\nu(x) = \sum_{n=0}^{\infty} \frac{(x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}$$

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)}$$

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$$

$$L_n^s(x) = \frac{1}{n!} x^{-s} e^x \frac{d^n}{dx^n} (x^{n+s} e^{-x})$$

$$P_n(x) = \frac{1}{n! 2^n} \frac{d^n}{dx^n} (x^2 - 1)^n$$

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

$$Se_{2n+p}(x, q) = \sum_{k=0}^{\infty} B_{2k+p}^{2n+p} \sinh[(2k+p)x]$$

$$se_{2n}(x, q) = \sum_{k=0}^{\infty} B_{2k}^{2n} \sin 2kx$$

$$se_{2n+1}(x, q) = \sum_{k=0}^{\infty} B_{2k+1}^{2n+1} \sin[(2k+1)x]$$

$$Y_\nu(x) = \frac{J_\nu(x) \cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$$

$$\gamma(\alpha, x) = \int_0^x e^{-\xi} \xi^{\alpha-1} d\xi$$

$$\Gamma(\alpha) = \int_0^\infty e^{-\xi} \xi^{\alpha-1} d\xi$$

$$\Phi(a, b; x) = 1 + \sum_{n=1}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}$$

even  $\pi$ -periodic Mathieu functions; these satisfy the equation  $y'' + (a - 2q \cos 2x)y = 0$ , where  $a = a_{2n}(q)$  are eigenvalues

even  $2\pi$ -periodic Mathieu functions; these satisfy the equation  $y'' + (a - 2q \cos 2x)y = 0$ , where  $a = a_{2n+1}(q)$  are eigenvalues

parabolic cylinder function (see Paragraph 7.3.4-1); it satisfies the equation  $y'' + (\nu + \frac{1}{2} - \frac{1}{4}x^2)y = 0$

error function

complementary error function

Hermite polynomial

Hankel function of first kind,  $i^2 = -1$

Hankel function of second kind,  $i^2 = -1$

hypergeometric function,  $(a)_n = a(a+1) \dots (a+n-1)$

modified Bessel function of first kind

Bessel function of first kind

modified Bessel function of second kind

generalized Laguerre polynomial

Legendre polynomial

associated Legendre functions

odd modified Mathieu functions, where  $p = 0, 1$ ;

$$Se_{2n+p}(x, q) = -i se_{2n+p}(ix, q)$$

odd  $\pi$ -periodic Mathieu functions; these satisfy the equation  $y'' + (a - 2q \cos 2x)y = 0$ , where  $a = b_{2n}(q)$  are eigenvalues

odd  $2\pi$ -periodic Mathieu functions; these satisfy the equation  $y'' + (a - 2q \cos 2x)y = 0$ , where  $a = b_{2n+1}(q)$  are eigenvalues

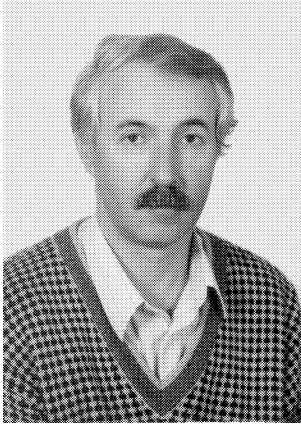
Bessel function of second kind

incomplete gamma function

gamma function

degenerate hypergeometric function,  
 $(a)_n = a(a+1) \dots (a+n-1)$

## AUTHOR



**Andrei D. Polyinin, D.Sc., Ph.D.**, is a noted scientist of broad interests, who works in various areas of mathematics, mechanics, and chemical engineering sciences.

A. D. Polyinin graduated from the Department of Mechanics and Mathematics of the Moscow State University in 1974. He received his Ph.D. degree in 1981 and D.Sc. degree in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, A. D. Polyinin has been a member of the staff of the Institute for Problems in Mechanics of the Russian Academy of Sciences.

Professor Polyinin has made important contributions to developing new exact and approximate analytical methods of the theory of differential equations, mathematical physics, integral equations, engineering mathematics, nonlinear mechanics, theory of heat and mass transfer, and chemical hydrodynamics. He obtained exact solutions for several thousand ordinary differential, partial differential, mathematical physics, and integral equations.

Professor Polyinin is an author of 27 books in English, Russian, German, and Bulgarian, as well as over 120 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyinin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, 1995; A. D. Polyinin and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, 1998; and A. D. Polyinin, V. F. Zaitsev, and A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Gordon and Breach, 2001.

In 1991, A. D. Polyinin was awarded a Chaplygin Prize of the USSR Academy of Sciences for his research in mechanics.

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3.3.2. Equations Containing Arbitrary Functions

3.3.3. Equations of the Form  $\rho(x, y, z) \frac{\partial w}{\partial t} = \text{div}[a(x, y, z)\nabla w] - q(x, y, z)w + \Phi(x, y, z, t)$

- 3.4. Equations with  $n$  Space Variables
  - 3.4.1. Equations of the Form  $\frac{\partial w}{\partial t} = a\Delta_n w + \Phi(x_1, \dots, x_n, t)$
  - 3.4.2. Other Equations Containing Arbitrary Parameters
  - 3.4.3. Equations Containing Arbitrary Functions

#### 4. Hyperbolic Equations with One Space Variable

- 4.1. Constant Coefficient Equations
  - 4.1.1. Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2}$
  - 4.1.2. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + \Phi(x, t)$
  - 4.1.3. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - bw + \Phi(x, t)$
  - 4.1.4. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} - b \frac{\partial w}{\partial x} + \Phi(x, t)$
  - 4.1.5. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + cw + \Phi(x, t)$
- 4.2. Wave Equation with Axial or Central Symmetry
  - 4.2.1. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right)$
  - 4.2.2. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$
  - 4.2.3. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right)$
  - 4.2.4. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) + \Phi(r, t)$
  - 4.2.5. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} \right) - bw + \Phi(r, t)$
  - 4.2.6. Equation of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \left( \frac{\partial^2 w}{\partial r^2} + \frac{2}{r} \frac{\partial w}{\partial r} \right) - bw + \Phi(r, t)$
- 4.3. Equations Containing Power Functions and Arbitrary Parameters
  - 4.3.1. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = (ax + b) \frac{\partial^2 w}{\partial x^2} + c \frac{\partial w}{\partial x} + kw + \Phi(x, t)$
  - 4.3.2. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = (ax^2 + b) \frac{\partial^2 w}{\partial x^2} + cx \frac{\partial w}{\partial x} + kw + \Phi(x, t)$
  - 4.3.3. Other Equations
- 4.4. Equations Containing the First Time Derivative
  - 4.4.1. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \frac{\partial^2 w}{\partial x^2} + b \frac{\partial w}{\partial x} + cw + \Phi(x, t)$
  - 4.4.2. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = f(x) \frac{\partial^2 w}{\partial x^2} + g(x) \frac{\partial w}{\partial x} + h(x)w + \Phi(x, t)$
  - 4.4.3. Other Equations
- 4.5. Equations Containing Arbitrary Functions
  - 4.5.1. Equations of the Form  $s(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} \left[ p(x) \frac{\partial w}{\partial x} \right] - q(x)w + \Phi(x, t)$
  - 4.5.2. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a(t) \frac{\partial w}{\partial t} = b(t) \left\{ \frac{\partial}{\partial x} \left[ p(x) \frac{\partial w}{\partial x} \right] - q(x)w \right\} + \Phi(x, t)$
  - 4.5.3. Other Equations

#### 5. Hyperbolic Equations with Two Space Variables

- 5.1. Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w$ 
  - 5.1.1. Problems in Cartesian Coordinates
  - 5.1.2. Problems in Polar Coordinates
  - 5.1.3. Axisymmetric Problems
- 5.2. Nonhomogeneous Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w + \Phi(x, y, t)$ 
  - 5.2.1. Problems in Cartesian Coordinates
  - 5.2.2. Problems in Polar Coordinates
  - 5.2.3. Axisymmetric Problems
- 5.3. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_2 w - bw + \Phi(x, y, t)$ 
  - 5.3.1. Problems in Cartesian Coordinates
  - 5.3.2. Problems in Polar Coordinates
  - 5.3.3. Axisymmetric Problems

5.4. Telegraph Equation  $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \Delta_2 w - bw + \Phi(x, y, t)$

5.4.1. Problems in Cartesian Coordinates

5.4.2. Problems in Polar Coordinates

5.4.3. Axisymmetric Problems

5.5. Other Equations with Two Space Variables

## 6. Hyperbolic Equations with Three or More Space Variables

6.1. Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w$

6.1.1. Problems in Cartesian Coordinates

6.1.2. Problems in Cylindrical Coordinates

6.1.3. Problems in Spherical Coordinates

6.2. Nonhomogeneous Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w + \Phi(x, y, z, t)$

6.2.1. Problems in Cartesian Coordinates

6.2.2. Problems in Cylindrical Coordinates

6.2.3. Problems in Spherical Coordinates

6.3. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_3 w - bw + \Phi(x, y, z, t)$

6.3.1. Problems in Cartesian Coordinates

6.3.2. Problems in Cylindrical Coordinates

6.3.3. Problems in Spherical Coordinates

6.4. Telegraph Equation  $\frac{\partial^2 w}{\partial t^2} + k \frac{\partial w}{\partial t} = a^2 \Delta_3 w - bw + \Phi(x, y, z, t)$

6.4.1. Problems in Cartesian Coordinates

6.4.2. Problems in Cylindrical Coordinates

6.4.3. Problems in Spherical Coordinates

6.5. Other Equations with Three Space Variables

6.5.1. Equations Containing Arbitrary Parameters

6.5.2. Equation of the Form  $\rho(x, y, z) \frac{\partial^2 w}{\partial t^2} = \operatorname{div} [a(x, y, z) \nabla w] - q(x, y, z)w + \Phi(x, y, z, t)$

6.6. Equations with  $n$  Space Variables

6.6.1. Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w$

6.6.2. Nonhomogeneous Wave Equation  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w + \Phi(x_1, \dots, x_n, t)$

6.6.3. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} = a^2 \Delta_n w - bw + \Phi(x_1, \dots, x_n, t)$

6.6.4. Equations Containing the First Time Derivative

## 7. Elliptic Equations with Two Space Variables

7.1. Laplace Equation  $\Delta_2 w = 0$

7.1.1. Problems in Cartesian Coordinate System

7.1.2. Problems in Polar Coordinate System

7.1.3. Other Coordinate Systems. Conformal Mappings Method

7.2. Poisson Equation  $\Delta_2 w = -\Phi(x)$

7.2.1. Preliminary Remarks. Solution Structure

7.2.2. Problems in Cartesian Coordinate System

7.2.3. Problems in Polar Coordinate System

7.2.4. Arbitrary Shape Domain. Conformal Mappings Method

7.3. Helmholtz Equation  $\Delta_2 w + \lambda w = -\Phi(x)$

7.3.1. General Remarks, Results, and Formulas

7.3.2. Problems in Cartesian Coordinate System

7.3.3. Problems in Polar Coordinate System

7.3.4. Other Orthogonal Coordinate Systems. Elliptic Domain

## 7.4. Other Equations

7.4.1. Stationary Schrödinger Equation  $\Delta_2 w = f(x, y)w$

### 7.4.2. Convective Heat and Mass Transfer Equations

### 7.4.3. Equations of Heat and Mass Transfer in Anisotropic Media

### 7.4.4. Other Equations Arising in Applications

7.4.5. Equations of the Form  $a(x)\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + b(x)\frac{\partial w}{\partial x} + c(x)w = -\Phi(x, y)$

## 8. Elliptic Equations with Three or More Space Variables

### 8.1. Laplace Equation $\Delta_3 w = 0$

#### 8.1.1. Problems in Cartesian Coordinates

#### 8.1.2. Problems in Cylindrical Coordinates

#### 8.1.3. Problems in Spherical Coordinates

#### 8.1.4. Other Orthogonal Curvilinear Systems of Coordinates

### 8.2. Poisson Equation $\Delta_3 w + \Phi(x) = 0$

#### 8.2.1. Preliminary Remarks. Solution Structure

#### 8.2.2. Problems in Cartesian Coordinates

#### 8.2.3. Problems in Cylindrical Coordinates

#### 8.2.4. Problems in Spherical Coordinates

### 8.3. Helmholtz Equation $\Delta_3 w + \lambda w = -\Phi(x)$

#### 8.3.1. General Remarks, Results, and Formulas

#### 8.3.2. Problems in Cartesian Coordinates

#### 8.3.3. Problems in Cylindrical Coordinates

#### 8.3.4. Problems in Spherical Coordinates

#### 8.3.5. Other Orthogonal Curvilinear Coordinates

### 8.4. Other Equations with Three Space Variables

#### 8.4.1. Equations Containing Arbitrary Functions

8.4.2. Equations of the Form  $\operatorname{div}[a(x, y, z)\nabla w] - q(x, y, z)w = -\Phi(x, y, z)$

### 8.5. Equations with $n$ Space Variables

8.5.1. Laplace Equation  $\Delta_n w = 0$

#### 8.5.2. Other Equations

## 9. Higher-Order Partial Differential Equations

### 9.1. Third-Order Partial Differential Equations

### 9.2. Fourth-Order One-Dimensional Nonstationary Equations

9.2.1. Equations of the Form  $\frac{\partial w}{\partial t} + a^2 \frac{\partial^4 w}{\partial x^4} = \Phi(x, t)$

9.2.2. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = 0$

9.2.3. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} = \Phi(x, t)$

9.2.4. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \frac{\partial^4 w}{\partial x^4} + kw = \Phi(x, t)$

#### 9.2.5. Other Equations

### 9.3. Two-Dimensional Nonstationary Fourth-Order Equations

9.3.1. Equations of the Form  $\frac{\partial w}{\partial t} + a^2 \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) = \Phi(x, y, t)$

9.3.2. Two-Dimensional Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w = 0$

9.3.3. Three- and  $n$ -Dimensional Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w = 0$

9.3.4. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \Delta \Delta w + kw = \Phi(x, y, t)$

9.3.5. Equations of the Form  $\frac{\partial^2 w}{\partial t^2} + a^2 \left( \frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} \right) + kw = \Phi(x, y, t)$

### 9.4. Fourth-Order Stationary Equations

9.4.1. Biharmonic Equation  $\Delta \Delta w = 0$

9.4.2. Equations of the Form  $\Delta \Delta w = \Phi(x, y)$

- 9.4.3. Equations of the Form  $\Delta \Delta w - \lambda w = \Phi(x, y)$
- 9.4.4. Equations of the Form  $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} = \Phi(x, y)$
- 9.4.5. Equations of the Form  $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^4 w}{\partial y^4} + kw = \Phi(x, y)$
- 9.4.6. Stokes Equation (Axisymmetric Flows of Viscous Fluids)
- 9.5. Higher-Order Linear Equations with Constant Coefficients
  - 9.5.1. Fundamental Solutions. Cauchy Problem
  - 9.5.2. Elliptic Equations
  - 9.5.3. Hyperbolic Equations
  - 9.5.4. Regular Equations. Number of Initial Conditions in the Cauchy Problem
  - 9.5.5. Some Special-Type Equations
- 9.6. Higher-Order Linear Equations with Variable Coefficients
  - 9.6.1. Equations Containing the First Time Derivative
  - 9.6.2. Equations Containing the Second Time Derivative
  - 9.6.3. Nonstationary Problems with Many Space Variables
  - 9.6.4. Some Special-Type Equations

## **Supplement A. Special Functions and Their Properties**

- A.1. Some Symbols and Coefficients
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  - A.1.2. Binomial Coefficients
  - A.1.3. Pochhammer Symbol
  - A.1.4. Bernoulli Numbers
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  - A.2.1. Error Function and Complementary Error Function
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- A.6. Bessel Functions
  - A.6.1. Definitions and Basic Formulas
  - A.6.2. Integral Representations and Asymptotic Expansions
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- A.9. Degenerate Hypergeometric Functions
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- A.16. Orthogonal Polynomials
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- B.8. Higher-Order Nonlinear Equations
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  - B.8.3. Other Equations

## References