HANDBOOK OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS for ENGINEERS and SCIENTISTS

Andrei D. Polyanin

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FOREWORD

Linear partial differential equations arise in various fields of science and numerous applications, e.g., heat and mass transfer theory, wave theory, hydrodynamics, aerodynamics, elasticity, acoustics, electrostatics, electrodynamics, electrical engineering, diffraction theory, quantum mechanics, control theory, chemical engineering sciences, and biomechanics.

This book presents brief statements and exact solutions of more than 2000 linear equations and problems of mathematical physics. Nonstationary and stationary equations with constant and variable coefficients of parabolic, hyperbolic, and elliptic types are considered. A number of new solutions to linear equations and boundary value problems are described. Special attention is paid to equations and problems of general form that depend on arbitrary functions. Formulas for the effective construction of solutions to nonhomogeneous boundary value problems of various types are given. We consider second-order and higher-order equations as well as the corresponding boundary value problems. All in all, the handbook presents more equations and problems of mathematical physics than any other book currently available.

For the reader's convenience, the introduction outlines some definitions and basic equations, problems, and methods of mathematical physics. It also gives useful formulas that enable one to express solutions to stationary and nonstationary boundary value problems of general form in terms of the Green's function.

Two supplements are given at the end of the book. Supplement A lists properties of the most common special functions (the gamma function, Bessel functions, degenerate hypergeometric functions, Mathieu functions, etc.). Supplement B describes the methods of generalized and functional separation of variables for nonlinear partial differential equations. We give specific examples and an overview application of these methods to construct exact solutions for various classes of second-, third-, fourth-, and higher-order equations (in total, about 150 nonlinear equations with solutions are described). Special attention is paid to equations of heat and mass transfer theory, wave theory, and hydrodynamics as well as to mathematical physics equations of general form that involve arbitrary functions.

The equations in all chapters are in ascending order of complexity. Many sections can be read independently, which facilitates working with the material. An extended table of contents will help the reader find the desired equations and boundary value problems. We refer to specific equations using notation like "1.8.5.2," which means "Equation 2 in Subsection 1.8.5."

To extend the range of potential readers with diverse mathematical backgrounds, the author strove to avoid the use of special terminology wherever possible. For this reason, some results are presented schematically, in a simplified manner (without details), which is however quite sufficient in most applications.

Separate sections of the book can serve as a basis for practical courses and lectures on equations of mathematical physics.

The author thanks Alexei Zhurov for useful remarks on the manuscript.

The author hopes that the handbook will be useful for a wide range of scientists, university teachers, engineers, and students in various areas of mathematics, physics, mechanics, control, and engineering sciences.

Andrei D. Polyanin

BASIC NOTATION

Latin Characters

$\mathscr{E}$	fundamental solution
$\mathrm{Im}[A]$	imaginary part of a complex quantity A
G	Green's function
$\mathbb{R}^n$	n-dimensional Euclidean space, $\mathbb{R}^n = \{-\infty < x_k < \infty;\ k = 1, \ldots, n\}$
$\mathrm{Re}[A]$	real part of a complex quantity A
$r,\ \varphi,\ z$	cylindrical coordinates, $r = \sqrt{x^2 + y^2}$ and $x = r\cos\varphi$, $y = r\sin\varphi$
$r,\ \theta,\ \varphi$	spherical coordinates, $r = \sqrt{x^2+y^2+z^2}$ and $x = r\sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = r\cos\theta$
t	time ($t \ge 0$)
w	unknown function (dependent variable)
x, y, z	space (Cartesian) coordinates
$x_1, \ldots, x_n$	Cartesian coordinates in n-dimensional space
$\mathbf{x}$	n-dimensional vector, $\mathbf{x} = \{x_1, \ldots, x_n\}$
$\lvert\mathbf{x}\rvert$	magnitude (length) of n-dimensional vector, $\lvert\mathbf{x}\rvert = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$
$\mathbf{y}$	n-dimensional vector, $\mathbf{y} = \{y_1, \ldots, y_n\}$

Greek Characters

Δ	Laplace operator
Δ_2	two-dimensional Laplace operator, $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
Δ_3	three-dimensional Laplace operator, $\Delta_3 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
Δ_n	n-dimensional Laplace operator, $\Delta_n = \sum\limits_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}$
$\delta(x)$	Dirac delta function; $\int_{-a}^{a} f(y)\delta(x-y)\,dy = f(x)$, where $f(x)$ is any continuous function, $a > 0$
δ_{nm}	Kronecker delta, $\delta_{nm} = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m \end{cases}$
$\vartheta(x)$	Heaviside unit step function, $\vartheta(x) = \begin{cases} 1 & \text{if } x \ge 0, \\ 0 & \text{if } x < 0 \end{cases}$

Brief Notation for Derivatives

$$\partial_t w = \frac{\partial w}{\partial t}, \quad \partial_x w = \frac{\partial w}{\partial x}, \quad \partial_{tt} w = \frac{\partial^2 w}{\partial t^2}, \quad \partial_{xx} w = \frac{\partial^2 w}{\partial x^2} \qquad \text{(partial derivatives)}$$

$$f'_x = \frac{df}{dx}, \quad f''_{xx} = \frac{d^2 f}{dx^2}, \quad f'''_{xxx} = \frac{d^3 f}{dx^3}, \quad f_x^{(n)} = \frac{d^n f}{dx^n} \qquad \text{(derivatives for } f = f(x))$$

Special Functions (See Also Supplement A)

$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt$	Airy function; $\mathrm{Ai}(x) = \frac{1}{\pi}\sqrt{\frac{1}{3}x}\, K_{1/3}\left(\frac{2}{3}x^{3/2}\right)$
$\mathrm{Ce}_{2n+p}(x, q) = \sum\limits_{k=0}^{\infty} A_{2k+p}^{2n+p} \cosh[(2k+p)x]$	even modified Mathieu functions, where $p = 0, 1$; $\mathrm{Ce}_{2n+p}(x, q) = \mathrm{ce}_{2n+p}(ix, q)$

$\mathrm{ce}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k}^{2n} \cos 2kx$ — even π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n}(q)$ are eigenvalues

$\mathrm{ce}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} A_{2k+1}^{2n+1} \cos[(2k+1)x]$ — even 2π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = a_{2n+1}(q)$ are eigenvalues

$D_\nu = D_\nu(x)$ — parabolic cylinder function (see Paragraph 7.3.4-1); it satisfies the equation $y'' + \left(\nu + \frac{1}{2} - \frac{1}{4}x^2\right)y = 0$

$\operatorname{erf} x = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x \exp\left(-\xi^2\right) d\xi$ — error function

$\operatorname{erfc} x = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_x^\infty \exp\left(-\xi^2\right) d\xi$ — complementary error function

$H_n(x) = (-1)^n e^{x^2} \dfrac{d^n}{dx^n}\left(e^{-x^2}\right)$ — Hermite polynomial

$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x)$ — Hankel function of first kind, $i^2 = -1$

$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x)$ — Hankel function of second kind, $i^2 = -1$

$F(a,b,c;x) = 1 + \sum\limits_{n=1}^{\infty} \dfrac{(a)_n (b)_n}{(c)_n} \dfrac{x^n}{n!}$ — hypergeometric function, $(a)_n = a(a+1)\ldots(a+n-1)$

$I_\nu(x) = \sum\limits_{n=0}^{\infty} \dfrac{(x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$ — modified Bessel function of first kind

$J_\nu(x) = \sum\limits_{n=0}^{\infty} \dfrac{(-1)^n (x/2)^{\nu+2n}}{n!\,\Gamma(\nu+n+1)}$ — Bessel function of first kind

$K_\nu(x) = \dfrac{\pi}{2} \dfrac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}$ — modified Bessel function of second kind

$L_n^s(x) = \dfrac{1}{n!} x^{-s} e^x \dfrac{d^n}{dx^n}\left(x^{n+s} e^{-x}\right)$ — generalized Laguerre polynomial

$P_n(x) = \dfrac{1}{n!\,2^n} \dfrac{d^n}{dx^n}(x^2-1)^n$ — Legendre polynomial

$P_n^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x)$ — associated Legendre functions

$\mathrm{Se}_{2n+p}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+p}^{2n+p} \sinh[(2k+p)x]$ — odd modified Mathieu functions, where $p = 0, 1$; $\mathrm{Se}_{2n+p}(x,q) = -i\,\mathrm{se}_{2n+p}(ix,q)$

$\mathrm{se}_{2n}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k}^{2n} \sin 2kx$ — odd π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = b_{2n}(q)$ are eigenvalues

$\mathrm{se}_{2n+1}(x,q) = \sum\limits_{k=0}^{\infty} B_{2k+1}^{2n+1} \sin[(2k+1)x]$ — odd 2π-periodic Mathieu functions; these satisfy the equation $y'' + (a - 2q\cos 2x)y = 0$, where $a = b_{2n+1}(q)$ are eigenvalues

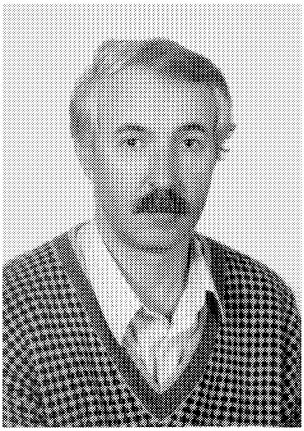
$Y_\nu(x) = \dfrac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}$ — Bessel function of second kind

$\gamma(\alpha,x) = \displaystyle\int_0^x e^{-\xi} \xi^{\alpha-1}\, d\xi$ — incomplete gamma function

$\Gamma(\alpha) = \displaystyle\int_0^\infty e^{-\xi} \xi^{\alpha-1}\, d\xi$ — gamma function

$\Phi(a,b;x) = 1 + \sum\limits_{n=1}^{\infty} \dfrac{(a)_n}{(b)_n} \dfrac{x^n}{n!}$ — degenerate hypergeometric function, $(a)_n = a(a+1)\ldots(a+n-1)$

AUTHOR



Andrei D. Polyanin, D.Sc., Ph.D., is a noted scientist of broad interests, who works in various areas of mathematics, mechanics, and chemical engineering sciences.

A. D. Polyanin graduated from the Department of Mechanics and Mathematics of the Moscow State University in 1974. He received his Ph.D. degree in 1981 and D.Sc. degree in 1986 at the Institute for Problems in Mechanics of the Russian (former USSR) Academy of Sciences. Since 1975, A. D. Polyanin has been a member of the staff of the Institute for Problems in Mechanics of the Russian Academy of Sciences.

Professor Polyanin has made important contributions to developing new exact and approximate analytical methods of the theory of differential equations, mathematical physics, integral equations, engineering mathematics, nonlinear mechanics, theory of heat and mass transfer, and chemical hydrodynamics. He obtained exact solutions for several thousand ordinary differential, partial differential, mathematical physics, and integral equations.

Professor Polyanin is an author of 27 books in English, Russian, German, and Bulgarian, as well as over 120 research papers and three patents. He has written a number of fundamental handbooks, including A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, 1995; A. D. Polyanin and A. V. Manzhirov, *Handbook of Integral Equations*, CRC Press, 1998; and A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, *Handbook of First Order Partial Differential Equations*, Gordon and Breach, 2001.

In 1991, A. D. Polyanin was awarded a Chaplygin Prize of the USSR Academy of Sciences for his research in mechanics.

Address: Institute for Problems in Mechanics, RAS, 101 Vernadsky Avenue, Building 1, 117526 Moscow, Russia
E-mail: polyanin@ipmnet.ru

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