CHAPTER 3

SHEARING FORCE AND BENDING MOMENT DIAGRAMS

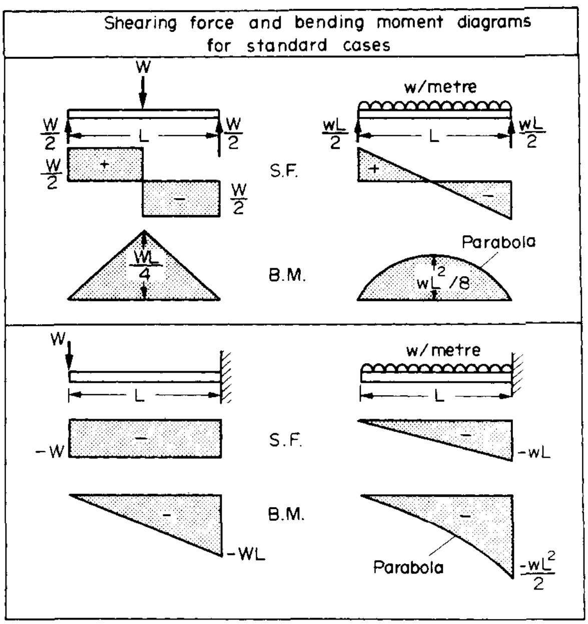
Summary

At any section in a beam carrying transverse loads the shearing force is defined as the algebraic sum of the forces taken on either side of the section.

Similarly, the bending moment at any section is the algebraic sum of the moments of the forces about the section, again taken on either side.

In order that the shearing-force and bending-moment values calculated on either side of the section shall have the same magnitude and sign, a convenient sign convention has to be adopted. This is shown in Figs. 3.1 and 3.2 (see page 42).

Shearing-force (S.F.) and bending-moment (B.M.) diagrams show the variation of these quantities along the length of a beam for any fixed loading condition.



3.1. Shearing force and bending moment

At every section in a beam carrying transverse loads there will be resultant forces on either side of the section which, for equilibrium, must be equal and opposite, and whose combined action tends to shear the section in one of the two ways shown in Fig. 3.1a and b. *The shearing force (S.F.) at the section is defined therefore as the algebraic sum of the forces taken on one side of the section.* Which side is chosen is purely a matter of convenience but in order that the value obtained on both sides shall have the same magnitude and sign a convenient sign convention has to be adopted.

3.1.1. Shearing force (S.F.) sign convention

Forces upwards to the left of a section or downwards to the right of the section are positive. Thus Fig. 3.1a shows a positive S.F. system at X–X and Fig. 3.1b shows a negative S.F. system.

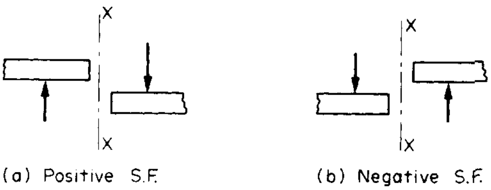


Fig. 3.1. S.F. sign convention.

In addition to the shear, every section of the beam will be subjected to bending, i.e. to a resultant B.M. which is the net effect of the moments of each of the individual loads. Again, for equilibrium, the values on either side of the section must have equal values. *The bending moment (B.M.) is defined therefore as the algebraic sum of the moments of the forces about the section, taken on either side of the section.* As for S.F., a convenient sign convention must be adopted.

3.1.2. Bending moment (B.M.) sign convention

Clockwise moments to the left and counterclockwise to the right are positive. Thus Fig. 3.2a shows a positive bending moment system resulting in *sagging* of the beam at X–X and Fig. 3.2b illustrates a negative B.M. system with its associated *hogging* beam.

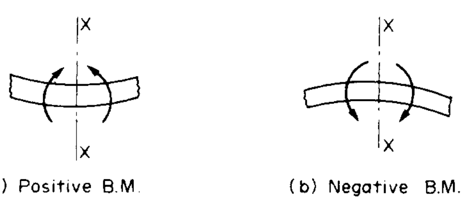


Fig. 3.2. B.M. sign convention.

It should be noted that whilst the above sign conventions for S.F. and B.M. are somewhat arbitrary and could be completely reversed, the systems chosen here are the only ones which yield the mathematically correct signs for slopes and deflections of beams in subsequent work and therefore are highly recommended.

Diagrams which illustrate the variation in the B.M. and S.F. values along the length of a beam or structure for any fixed loading condition are termed *B.M. and S.F. diagrams*. They are therefore graphs of B.M. or S.F. values drawn on the beam as a base and they clearly illustrate in the early design stages the positions on the beam which are subjected to the greatest shear or bending stresses and hence which may require further consideration or strengthening.

At this point it is imperative to note that there are two general forms of loading to which structures may be subjected, namely, concentrated and distributed loads. The former are assumed to act at a point and immediately introduce an oversimplification since all practical loading systems must be applied over a finite area. Nevertheless, for calculation purposes this area is assumed to be so small that the load can be justly assumed to act at a point. Distributed loads are assumed to act over part, or all, of the beam and in most cases are assumed to be equally or uniformly distributed; they are then termed uniformly distributed loads (u.d.l.). Occasionally, however, the distribution is not uniform but may vary linearly across the loaded portion or have some more complex distribution form.

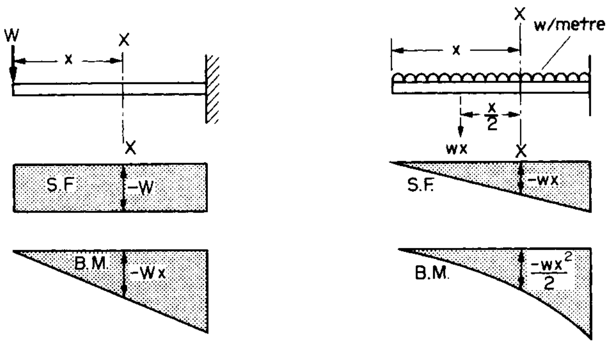


Fig. 3.3. S.F.–B.M. diagrams for standard cases.

Thus in the case of a cantilever carrying a concentrated load W at the end (Fig. 3.3), the S.F. at any section X–X, distance x from the free end, is S.F. $= -W$. This will be true whatever the value of x, and so the S.F. diagram becomes a rectangle. The B.M. at the same section X–X is $-Wx$ and this will increase linearly with x. The B.M. diagram is therefore a triangle.

If the cantilever now carries a uniformly distributed load, the S.F. at X–X is the net load to one side of X–X, i.e. $-wx$. In this case, therefore, the S.F. diagram becomes triangular, increasing to a maximum value of $-wL$ at the support. The B.M. at X–X is obtained by treating the load to the left of X–X as a concentrated load of the same value acting at the centre of gravity,

i.e. $$\text{B.M. at } X\text{–}X = -wx\frac{x}{2} = -\frac{-wx^2}{2}$$

Plotted against x this produces the parabolic B.M. diagram shown.

3.2. S.F. and B.M. diagrams for beams carrying concentrated loads only

In order to illustrate the procedure to be adopted for the determination of S.F. and B.M. values for more complicated load conditions, consider the simply supported beam shown in

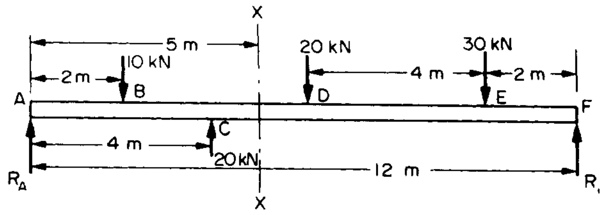


Fig. 3.4.

Fig. 3.4 carrying concentrated loads only. (The term *simply supported* means that the beam can be assumed to rest on knife-edges or roller supports and is free to bend at the supports without any restraint.)

The values of the reactions at the ends of the beam may be calculated by applying normal equilibrium conditions, i.e. by taking moments about F.

Thus
$$R_A \times 12 = (10 \times 10) + (20 \times 6) + (30 \times 2) - (20 \times 8) = 120$$
$$R_A = 10\,\text{kN}$$

For vertical equilibrium

$$\text{total force up} = \text{total load down}$$
$$R_A + R_F = 10 + 20 + 30 - 20 = 40$$
$$R_F = 30\,\text{kN}$$

At this stage it is advisable to check the value of R_F by taking moments about A.

Summing up the forces on either side of X–X we have the result shown in Fig. 3.5. Using the sign convention listed above, the shear force at X–X is therefore $+20$ kN, i.e. the resultant force at X–X tending to shear the beam is 20 kN.

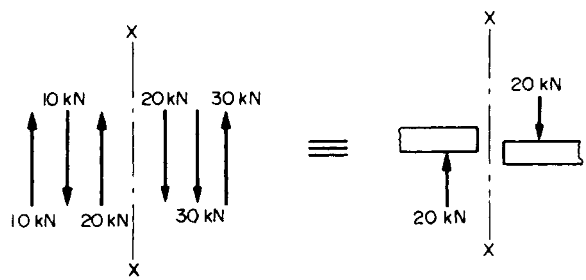


Fig. 3.5. Total S.F. at X–X.

Similarly, Fig. 3.6 shows the summation of the moments of the forces at X–X, the resultant B.M. being 40 kN m.

In practice only one side of the section is normally considered and the summations involved can often be completed by mental arithmetic. The complete S.F. and B.M. diagrams for the beam are shown in Fig. 3.7, and the B.M. values used to construct the diagram are derived on page 45.

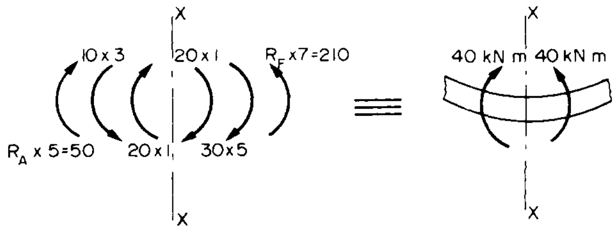


Fig. 3.6. Total B.M. at X–X.

$$\begin{aligned}
&\text{B.M. at } A && = 0\\
&\text{B.M. at } B = +(10 \times 2) && = +20\,\text{kN m}\\
&\text{B.M. at } C = +(10 \times 4) - (10 \times 2) && = +20\,\text{kN m}\\
&\text{B.M. at } D = +(10 \times 6) + (20 \times 2) - (10 \times 4) && = +60\,\text{kN m}\\
&\text{B.M. at } E = +(30 \times 2) && = +60\,\text{kN m}\\
&\text{B.M. at } F && = 0
\end{aligned}$$

All the above values have been calculated from the moments of the forces to the *left* of each section considered except for E where forces to the right of the section are taken.

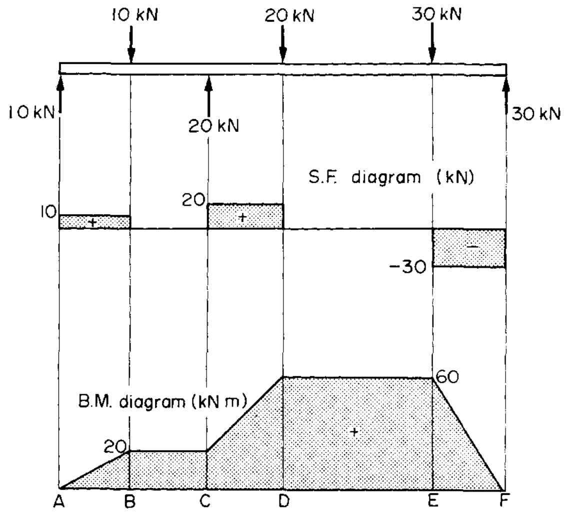


Fig. 3.7.

It may be observed at this stage that the S.F. diagram can be obtained very quickly when working from the left-hand side, since after plotting the S.F. value at the support all subsequent steps are in the direction of and equal in magnitude to the applied loads, e.g. 10 kN up at A, down 10 kN at B, up 20 kN at C, etc., with horizontal lines joining the steps to show that the S.F. remains constant between points of application of concentrated loads.

The S.F. and B.M. values at the left-hand support are determined by considering a section an infinitely small distance to the right of the support. The only load to the left (and hence the

S.F.) is then the reaction of 10 kN upwards, i.e. positive, and the bending moment = reaction × zero distance = zero.

The following characteristics of the two diagrams are now evident and will be explained later in this chapter:

(a) between B and C the S.F. is zero and the B.M. remains constant;
(b) between A and B the S.F. is positive and the slope of the B.M. diagram is positive; vice versa between E and F;
(c) the difference in B.M. between A and B = 20 kN m = area of S.F. diagram between A and B.

3.3. S.F. and B.M. diagrams for uniformly distributed loads

Consider now the simply supported beam shown in Fig. 3.8 carrying a u.d.l. $w = 25$ kN/m across the complete span.

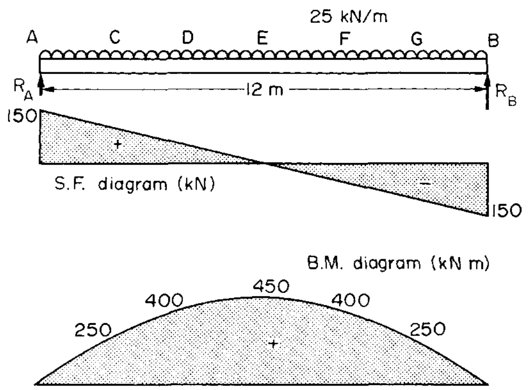


Fig. 3.8.

Here again it is necessary to evaluate the reactions, but in this case the problem is simplified by the symmetry of the beam. Each reaction will therefore take half the applied load,

i.e.
$$R_A = R_B = \frac{25 \times 12}{2} = 150\,\text{kN}$$

The S.F. at A, using the usual sign convention, is therefore $+150$ kN.

Consider now the beam divided into six equal parts 2 m long. The S.F. at any other point C is, therefore,

$$150 - \text{load downwards between } A \text{ and } C$$
$$= 150 - (25 \times 2) = +100\,\text{kN}$$

The whole diagram may be constructed in this way, or much more quickly by noticing that the S.F. at A is $+150$ kN and that between A and B the S.F. decreases uniformly, producing the required sloping straight line, shown in Fig. 3.7. Alternatively, the S.F. at A is $+150$ kN and between A and B this decreases gradually by the amount of the applied load (i.e. by $25 \times 12 = 300$ kN) to -150 kN at B.

When evaluating B.M.'s it is assumed that a u.d.l. can be replaced by a concentrated load of equal value acting at the middle of its spread. When taking moments about C, therefore, the portion of the u.d.l. between A and C has an effect equivalent to that of a concentrated load of $25 \times 2 = 50$ kN acting the centre of AC, i.e. 1 m from C.

$$\text{B.M. at } C = (R_A \times 2) - (50 \times 1) = 300 - 50 = 250 \text{ kN m}$$

Similarly, for moments at D the u.d.l. on AD can be replaced by a concentrated load of

$$25 \times 4 = 100 \text{ kN at the centre of } AD, \text{ i.e. at } C.$$

$$\text{B.M. at } D = (R_A \times 4) - (100 \times 2) = 600 - 200 = 400 \text{ kN m}$$

Similarly,

$$\text{B.M. at } E = (R_A \times 6) - (25 \times 6)3 = 900 - 450 = 450 \text{ kN m}$$

The B.M. diagram will be symmetrical about the beam centre line; therefore the values of B.M. at F and G will be the same as those at D and C respectively. The final diagram is therefore as shown in Fig. 3.8 and is parabolic.

Point (a) of the summary is clearly illustrated here, since the B.M. is a maximum when the S.F. is zero. Again, the reason for this will be shown later.

3.4. S.F. and B.M. diagrams for combined concentrated and uniformly distributed loads

Consider the beam shown in Fig. 3.9 loaded with a combination of concentrated loads and u.d.l.s.

Taking moments about E

$$(R_A \times 8) + (40 \times 2) = (10 \times 2 \times 7) + (20 \times 6) + (20 \times 3) + (10 \times 1) + (20 \times 3 \times 1.5)$$

$$8R_A + 80 = 420$$

$$R_A = 42.5 \text{ kN} (= \text{S.F. at } A)$$

Now

$$R_A + R_E = (10 \times 2) + 20 + 20 + 10 + (20 \times 3) + 40 = 170$$

$$R_E = 127.5 \text{ kN}$$

Working from the left-hand support it is now possible to construct the S.F. diagram, as indicated previously, by following the direction arrows of the loads. In the case of the u.d.l.'s the S.F. diagram will decrease gradually by the amount of the total load until the end of the u.d.l. or the next concentrated load is reached. Where there is no u.d.l. the S.F. diagram remains horizontal between load points.

In order to plot the B.M. diagram the following values must be determined:

$$\text{B.M. at } A = 0$$

$$\text{B.M. at } B = (42.5 \times 2) - (10 \times 2 \times 1) = 85 - 20 = 65 \text{ kN m}$$

$$\text{B.M. at } C = (42.5 \times 5) - (10 \times 2 \times 4) - (20 \times 3) = 212.5 - 80 - 60 = 72.5 \text{ kN m}$$

$$\text{B.M. at } D = (42.5 \times 7) - (10 \times 2 \times 6) - (20 \times 5) - (20 \times 2) - (20 \times 2 \times 1) = 297.5 - 120 - 100 - 40 - 40 = 297.5 - 300 = -2.5 \text{ kN m}$$

$$\text{B.M. at } E = (-40 \times 2) \text{ working from r.h.s.} = -80 \text{ kN m}$$

$$\text{B.M. at } F = 0$$

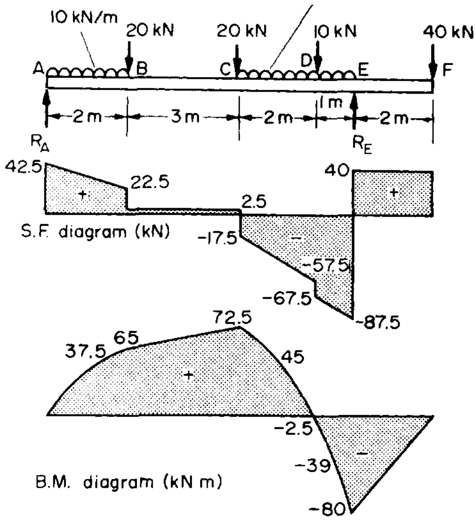


Fig. 3.9.

For complete accuracy one or two intermediate values should be obtained along each u.d.l. portion of the beam,

e.g. B.M. midway between A and $B = (42.5 \times 1) - (10 \times 1 \times \frac{1}{2})$

$$= 42.5 - 5 = 37.5 \text{ kN m}$$

Similarly, B.M. midway between C and $D = 45$ kN m

B.M. midway between D and $E = -39$ kN m

The B.M. and S.F. diagrams are then as shown in Fig. 3.9.

3.5. Points of contraflexure

A point of contraflexure is a point where the curvature of the beam changes sign. It is sometimes referred to as a *point of inflexion* and will be shown later to occur at the point, or points, on the beam where the B.M. is zero.

For the beam of Fig. 3.9, therefore, it is evident from the B.M. diagram that this point lies somewhere between C and D (B.M. at C is positive, B.M. at D is negative). If the required point is a distance x from C then at that point

$$\text{B.M.} = (42.5)(5+x) - (10 \times 2)(4+x) - 20(3+x) - 20x - \frac{20x^2}{2}$$

$$= 212.5 + 42.5x - 80 - 20x - 60 - 20x - 20x - 10x^2$$

$$= 72.5 - 17.5x - 10x^2$$

Thus the B.M. is zero where

$$0 = 72.5 - 17.5x - 10x^2$$

i.e. where $x = 1.96$ or -3.7

Since the last answer can be ignored (being outside the beam), the point of contraflexure must be situated at 1.96 m to the right of C.

3.6. Relationship between shear force Q, bending moment M and intensity of loading w

Consider the beam AB shown in Fig. 3.10 carrying a uniform loading intensity (uniformly distributed load) of w kN/m. By symmetry, each reaction takes half the total load, i.e., $wL/2$.

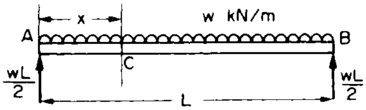


Fig. 3.10.

The B.M. at any point C, distance x from A, is given by

$$M = \frac{wL}{2}x - (wx)\frac{x}{2}$$

i.e.

$$M = \tfrac{1}{2}wLx - \tfrac{1}{2}wx^2$$

Differentiating,

$$\frac{dM}{dx} = \tfrac{1}{2}wL - wx$$

Now

$$\text{S.F. at } C = \tfrac{1}{2}wL - wx = Q \qquad (3.1)$$

$\therefore$

$$\frac{dM}{dx} = Q \qquad (3.2)$$

Differentiating eqn. (3.1),

$$\frac{dQ}{dx} = -w \qquad (3.3)$$

These relationships are the basis of the rules stated in the summary, the proofs of which are as follows:

(a) The maximum or minimum B.M. occurs where $dM/dx = 0$

But

$$\frac{dM}{dx} = Q$$

Thus where S.F. is zero B.M. is a maximum or minimum.

(b) The slope of the B.M. diagram $= dM/dx = Q$.

Thus where $Q = 0$ the slope of the B.M. diagram is zero, and the B.M. is therefore constant.

(c) Also, since Q represents the slope of the B.M. diagram, it follows that ***where the S.F. is positive the slope of the B.M. diagram is positive, and where the S.F. is negative the slope of the B.M. diagram is also negative.***

(d) The area of the S.F. diagram between any two points, from basic calculus, is

$$\int Q\,dx$$

But
$$\frac{dM}{dx} = Q \quad \text{or} \quad M = \int Q dx$$

i.e. ***the B.M. change between any two points is the area of the S.F. diagram between these points.***

This often provides a very quick method of obtaining the B.M. diagram once the S.F. diagram has been drawn.

(e) With the chosen sign convention, when the B.M. is positive the beam is *sagging* and when it is negative the beam is *hogging*. Thus when the curvature of the beam changes from *sagging* to *hogging*, as at X–X in Fig. 3.11, or vice versa, the B.M. changes sign, i.e. becomes instantaneously zero. This is termed a ***point of inflexion*** or ***contraflexure. Thus a point of contraflexure occurs where the B.M. is zero.***

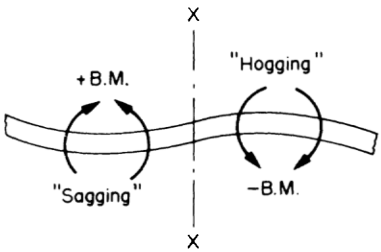


Fig. 3.11. Beam with point of contraflexure at X–X.

3.7. S.F. and B.M. diagrams for an applied couple or moment

In general there are two ways in which the couple or moment can be applied: (a) with horizontal loads and (b) with vertical loads, and the method of solution is different for each.

Type (a): couple or moment applied with horizontal loads

Consider the beam AB shown in Fig. 3.12 to which a moment $F.d$ is applied by means of horizontal loads at a point C, distance a from A.

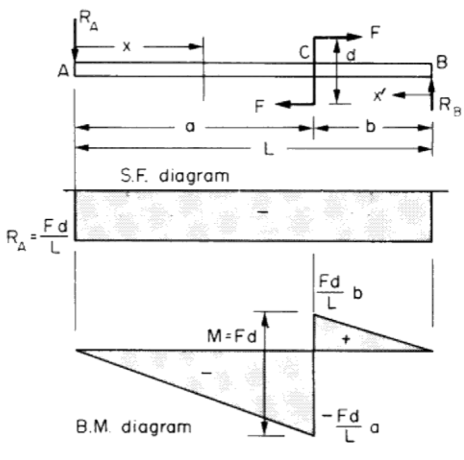


Fig. 3.12.

Since this will tend to lift the beam at A, R_A acts downwards.

Moments about B: $$R_A \times L = Fd$$

$$\therefore \quad R_A = \frac{Fd}{L}$$

and for vertical equilibrium $$R_B = R_A = \frac{Fd}{L}$$

The S.F. diagram can now be drawn as the horizontal loads have no effect on the vertical shear.

The B.M. at any section between A and C is

$$M = -R_A x = \frac{-Fd}{L}x$$

Thus the value of the B.M. increases linearly from zero at A to $\frac{-Fd}{L}a$ at C.

Similarly, the B.M. at any section between C and B is

$$M = -R_A x + Fd = R_B x' = \frac{Fd}{L}x'$$

i.e. the value of the B.M. again increases linearly from zero at B to $\frac{Fd}{L}b$ at C. The B.M. diagram is therefore as shown in Fig. 3.12.

Type (b): moment applied with vertical loads

Consider the beam AB shown in Fig. 3.13; taking moments about B:

$$R_A L = F(d+b)$$

$$\therefore \quad R_A = \frac{F(d+b)}{L}$$

Similarly, $$R_B = \frac{F(a-d)}{L}$$

The S.F. diagram can therefore be drawn as in Fig. 3.13 and it will be observed that in this case F does affect the diagram.

For the B.M. diagram an equivalent system is used. The offset load F is replaced by a moment and a force acting at C, as shown in Fig. 3.13. Thus

$$\text{B.M. between } A \text{ and } C = R_A x$$

$$= \frac{F(d+b)}{L}x$$

i.e. increasing linearly from zero to $\frac{F(d+b)}{L}a$ at C.

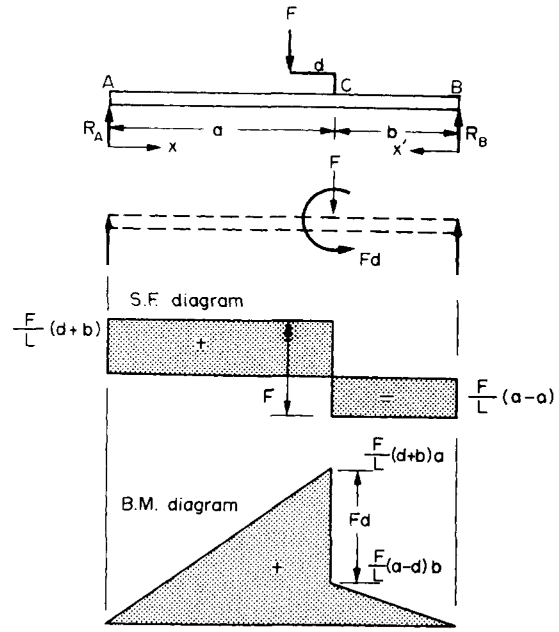


Fig. 3.13.

Similarly,

$$\text{B.M. between } C \text{ and } B = R_B x'$$

$$= \frac{F(a-d)}{L} x'$$

i.e. increasing linearly from zero to $\dfrac{F(a-d)}{L} b$ at C.

The difference in values at C is equal to the applied moment Fd, as with type (a).

Consider now the beam shown in Fig. 3.14 carrying concentrated loads in addition to the applied moment of 30 kN m (which can be assumed to be of type (a) unless otherwise stated). The principle of superposition states that the total effect of the combined loads will be the same as the algebraic sum of the effects of the separate loadings, i.e. the final diagram will be the combination of the separate diagrams representing applied moment and those representing concentrated loads. The final diagrams are therefore as shown shaded, all values quoted being measured from the normal base line of each diagram. In each case, however, the applied-moment diagrams have been inverted so that the negative areas can easily be subtracted. Final values are now measured from the dotted lines: e.g. the S.F. and B.M. at any point G are as indicated in Fig. 3.14.

3.8. S.F. and B.M. diagrams for inclined loads

If a beam is subjected to inclined loads as shown in Fig. 3.15 each of the loads must be resolved into its vertical and horizontal components as indicated. The vertical components

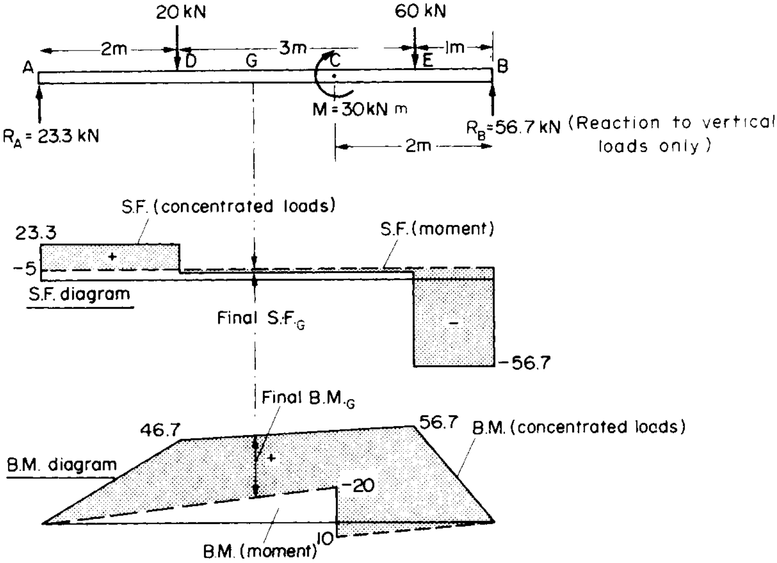


Fig. 3.14.

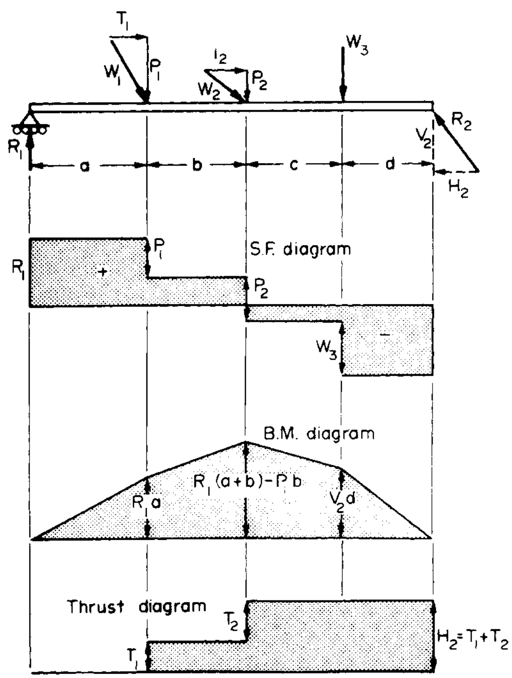


Fig. 3.15. S.F., B.M. and thrust diagrams for system of inclined loads.

yield the values of the vertical reactions at the supports and hence the S.F. and B.M. diagrams are obtained as described in the preceding sections. In addition, however, there must be a horizontal constraint applied to the beam at one or both reactions to bring the horizontal components of the applied loads into equilibrium. Thus there will be a horizontal force or *thrust diagram* for the beam which indicates the axial load carried by the beam at any point. If the constraint is assumed to be applied at the right-hand end the thrust diagram will be as indicated.

3.9. Graphical construction of S.F. and B.M. diagrams

Consider the simply supported beam shown in Fig. 3.16 carrying three concentrated loads of different values. The procedure to be followed for graphical construction of the S.F. and B.M. diagrams is as follows.

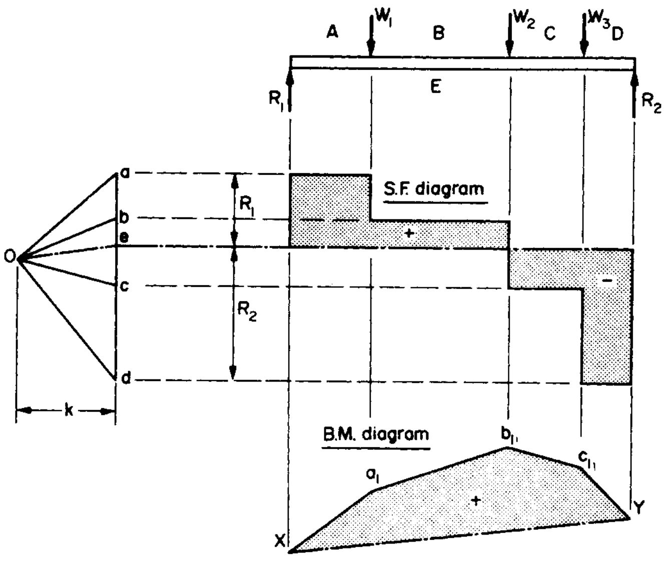


Fig. 3.16. Graphical construction of S.F. and B.M. diagrams.

(a) Letter the spaces between the loads and reactions A, B, C, D and E. Each force can then be denoted by the letters of the spaces on either side of it.
(b) To one side of the beam diagram construct a force vector diagram for the applied loads, i.e. set off a vertical distance *ab* to represent, in magnitude and direction, the force W_1 dividing spaces A and B to some scale, *bc* to represent W_2 and *cd* to represent W_3.
(c) Select any point O, known as a *pole point*, and join *Oa*, *Ob*, *Oc* and *Od*.
(d) Drop verticals from all loads and reactions.
(e) Select any point X on the vertical through reaction R_1 and from this point draw a line in space A parallel to *Oa* to cut the vertical through W_1 in a_1. In space B draw a line from a_1 parallel to *ob*, continue in space C parallel to *Oc*, and finally in space D parallel to *Od* to cut the vertical through R_2 in Y.

(f) Join XY and through the pole point O draw a line parallel to XY to cut the force vector diagram in e. The distance ea then represents the value of the reaction R_1 in magnitude and direction and de represents R_2.
(g) Draw a horizontal line through e to cut the vertical projections from the loading points and to act as the base line for the S.F. diagram. Horizontal lines from a in gap A, b in gap B, c in gap C, etc., produce the required S.F. diagram to the same scale as the original force vector diagram.
(h) The diagram $Xa_1b_1c_1Y$ is the B.M. diagram for the beam, vertical distances from the inclined base line XY giving the bending moment at any required point to a certain scale.

If the original beam diagram is drawn to a scale 1 cm = L metres (say), the force vector diagram scale is 1 cm = W newton, and, if the horizontal distance from the pole point O to the vector diagram is k cm, then the scale of the B.M. diagram is

$$1\ \text{cm} = kLW\ \text{newton metre}$$

The above procedure applies for beams carrying concentrated loads only, but an approximate solution is obtained in a similar way for u.d.l.s. by considering the load divided into a convenient number of concentrated loads acting at the centres of gravity of the divisions chosen.

3.10. S.F. and B.M. diagrams for beams carrying distributed loads of increasing value

For beams which carry distributed loads of varying intensity as in Fig. 3.18 a solution can be obtained from eqn. (3.3) provided that the loading variation can be expressed in terms of the distance x along the beam span, i.e. as a function of x.

$$\frac{dQ}{dx} = -w = -f(x)$$

Integrating once yields the shear force Q in terms of a constant of integration A since

$$\frac{dM}{dx} = Q$$

Integration again yields an expression for the B.M. M in terms of A and a second constant of integration B. Known conditions of B.M. or S.F., usually at the supports or ends of the beam, yield the values of the constants and hence the required distributions of S.F. and B.M. A typical example of this type has been evaluated on page 57.

3.11. S.F. at points of application of concentrated loads

In the preceding sections it has been assumed that concentrated loads can be applied precisely at a point so that S.F. diagrams are shown to change value suddenly from one value to another, and sometimes one sign to another, at the loading points. It would appear from the S.F. diagrams drawn previously, therefore, that two possible values of S.F. exist at any one loading point and this is obviously not the case. In practice, loads can only be applied over finite areas and the S.F. must change gradually from one value to another across these areas. The vertical line portions of the S.F. diagrams are thus highly idealised versions of what actually occurs in practice and should be replaced more accurately by lines slightly inclined to the vertical. All sharp corners of the diagrams should also be rounded. Despite these minor inaccuracies, B.M. and S.F. diagrams remain a highly convenient, powerful and useful representation of beam loading conditions for design purposes.

Examples

Example 3.1

Draw the S.F. and B.M. diagrams for the beam loaded as shown in Fig. 3.17, and determine (a) the position and magnitude of the maximum B.M., and (b) the position of any point of contraflexure.

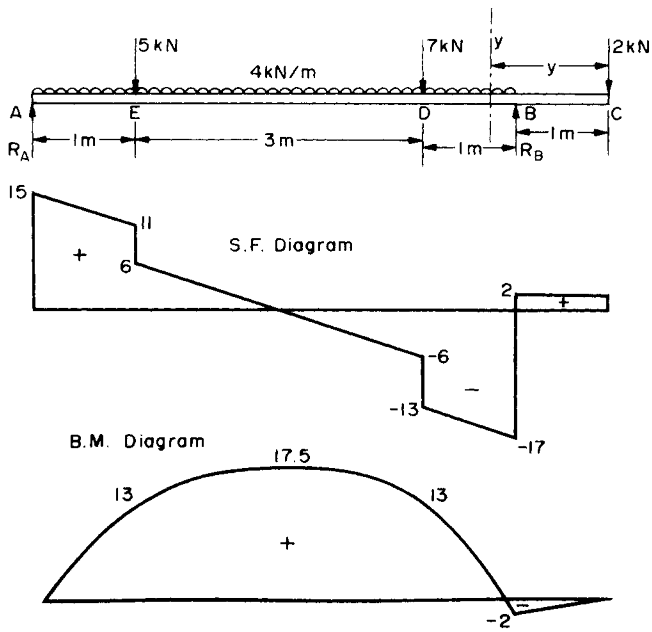


Fig. 3.17.

Solution

Taking the moments about A,

$$5R_B = (5 \times 1) + (7 \times 4) + (2 \times 6) + (4 \times 5) \times 2.5$$

$$\therefore \qquad R_B = \frac{5 + 28 + 12 + 50}{5} = \mathbf{19\,kN}$$

and since $$R_A + R_B = 5+7+2+(4\times 5) = 34$$
$$R_A = 34-19 = \mathbf{15\,kN}$$

The S.F. diagram may now be constructed as described in §3.4 and is shown in Fig. 3.17.

Calculation of bending moments

B.M. at A and $C = 0$

B.M. at B $= -2\times 1 = -2\,\text{kN m}$

B.M. at D $= -(2\times 2)+(19\times 1)-(4\times 1\times \frac{1}{2}) = +13\,\text{kN m}$

B.M. at E $= +(15\times 1)-(4\times 1\times \frac{1}{2}) = +13\,\text{kN m}$

The maximum B.M. will be given by the point (or points) at which dM/dx (i.e. the shear force) is zero. By inspection of the S.F. diagram this occurs midway between D and E, i.e. at 1.5 m from E.

$$\text{B.M. at this point} = (2.5\times 15)-(5\times 1.5)-\left(4\times 2.5\times\frac{2.5}{2}\right)$$
$$= \mathbf{+17.5\,kN\,m}$$

There will also be local maxima at the other points where the S.F. diagram crosses its zero axis, i.e. at point B.

Owing to the presence of the concentrated loads (reactions) at these positions, however, these will appear as discontinuities in the diagram; there will not be a smooth contour change. The value of the B.M.s at these points should be checked since the position of maximum stress in the beam depends upon the numerical maximum value of the B.M.; this does not necessarily occur at the mathematical maximum obtained above.

The B.M. diagram is therefore as shown in Fig. 3.17. Alternatively, the B.M. at any point between D and E at a distance of x from A will be given by

$$M_{xx} = 15x - 5(x-1) - \frac{4x^2}{2} = 10x + 5 - 2x^2$$

The maximum B.M. position is then given where $\dfrac{dM}{dx} = 0.$

$$\frac{dM}{dx} = 10 - 4x = 0 \qquad \therefore \qquad x = 2.5\,\text{m}$$

i.e. **1.5 m from E,** as found previously.

(b) Since the B.M. diagram only crosses the zero axis once there is only one point of contraflexure, i.e. between B and D. Then, B.M. at distance y from C will be given by

$$M_{yy} = -2y + 19(y-1) - 4(y-1)\tfrac{1}{2}(y-1)$$
$$= -2y + 19y - 19 - 2y^2 + 4y - 2 = 0$$

The point of contraflexure occurs where B.M. = 0, i.e. where $M_{yy} = 0$,

$$\therefore \qquad 0 = -2y^2 + 21y - 21$$

i.e. $$2y^2 - 21y + 21 = 0$$

Then $$y = \frac{21 \pm \sqrt{(21^2 - 4 \times 2 \times 21)}}{4} = 1.12\,\text{m}$$

i.e. point of contraflexure occurs **0.12 m to the left of *B*.**

Example 3.2

A beam *ABC* is 9 m long and supported at *B* and *C*, 6 m apart as shown in Fig. 3.18. The beam carries a triangular distribution of load over the portion *BC* together with an applied counterclockwise couple of moment 80 kN m at *B* and a u.d.l. of 10 kN/m over *AB*, as shown. Draw the S.F. and B.M. diagrams for the beam.

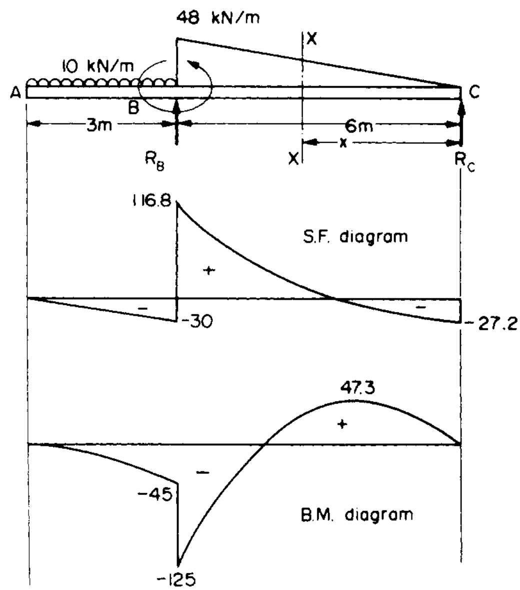


Fig. 3.18.

Solution

Taking moments about *B*,

$$(R_C \times 6) + (10 \times 3 \times 1.5) + 80 = (\tfrac{1}{2} \times 6 \times 48) \times \tfrac{1}{3} \times 6$$
$$6R_C + 45 + 80 = 288$$
$$R_C = 27.2\,\text{kN}$$

and $$R_C + R_B = (10 \times 3) + (\tfrac{1}{2} \times 6 \times 48)$$
$$= 30 + 144 = 174$$

$\therefore$ $$R_B = 146.8\,\text{kN}$$

At any distance x from C between C and B the shear force is given by

$$\text{S.F.}_{XX} = -\tfrac{1}{2}wx + R_C$$

and by proportions
$$\frac{w}{x} = \frac{48}{6} = 8$$

i.e.
$$w = 8x \text{ kN/m}$$

$\therefore$
$$\begin{aligned} \text{S.F.}_{XX} &= -(R_C - \tfrac{1}{2} \times 8x \times x) \\ &= -R_C + 4x^2 \\ &= -27.2 + 4x^2 \end{aligned}$$

The S.F. diagram is then as shown in Fig. 3.18.

Also
$$\begin{aligned} \text{B.M.}_{XX} &= -(\tfrac{1}{2}wx)\frac{x}{3} + R_C x \\ &= 27.2x - \frac{4x^3}{3} \end{aligned}$$

For a maximum value,
$$\frac{d\,(\text{B.M.})}{dx} = \text{S.F.} = 0$$

i.e.,where
$$4x^2 = 27.2$$

or
$$x = 2.61 \text{ m from } C$$

$$\begin{aligned} \text{B.M.}_{max} &= 27.2(2.61) - \frac{4}{3}(2.61)^3 \\ &= 47.3 \text{ kN m} \end{aligned}$$

$$\text{B.M. at } A \text{ and } C = 0$$

$$\text{B.M. immediately to left of } B = -(10 \times 3 \times 1.5) = -45 \text{ kN m}$$

At the point of application of the applied moment there will be a sudden change in B.M. of 80 kN m. (There will be no such discontinuity in the S.F. diagram; the effect of the moment will merely be reflected in the values calculated for the reactions.)

The B.M. diagram is therefore as shown in Fig. 3.18.

Problems

3.1 (A). A beam AB, 1.2 m long, is simply-supported at its ends A and B and carries two concentrated loads, one of 10 kN at C, the other 15 kN at D. Point C is 0.4 m from A, point D is 1 m from A. Draw the S.F. and B.M. diagrams for the beam inserting principal values. [9.17, −0.83, −15.83 kN; 3.67, 3.17 kNm.]

3.2 (A). The beam of question 3.1 carries an additional load of 5 kN *upwards* at point E, 0.6 m from A. Draw the S.F. and B.M. diagrams for the modified loading. What is the maximum B.M.?
[6.67, −3.33, 1.67, −13.33 kN; 2.67, 2, 2.67 kN m.]

3.3 (A). A cantilever beam AB, 2.5 m long is rigidly built in at A and carries vertical concentrated loads of 8 kN at B and 12 kN at C, 1 m from A. Draw S.F. and B.M. diagrams for the beam inserting principal values.
[−8, −20 kN; −11.2, −31.2 kNm.]

3.4 (A). A beam AB, 5 m long, is simply-supported at the end B and at a point C, 1 m from A. It carries vertical loads of 5 kN at A and 20 kN at D, the centre of the span BC. Draw S.F. and B.M. diagrams for the beam inserting principal values. [−5, 11.25, −8.75 kN; −5, 17.5 kNm.]

3.5 (A). A beam AB, 3 m long, is simply-supported at A and B. It carries a 16 kN concentrated load at C, 1.2 m from A, and a u.d.l. of 5 kN/m over the remainder of the beam. Draw the S.F. and B.M. diagrams and determine the value of the maximum B.M. [12.3, −3.7, −12.7 kN; 14.8 kNm.]

3.6 (A). A simply supported beam has a span of 4 m and carries a uniformly distributed load of 60 kN/m together with a central concentrated load of 40 kN. Draw the S.F. and B.M. diagrams for the beam and hence determine the maximum B.M. acting on the beam. [S.F. 140, ±20, −140 kN; B.M. 0, 160, 0 kN m.]

3.7 (A). A 2 m long cantilever is built-in at the right-hand end and carries a load of 40 kN at the free end. In order to restrict the deflection of the cantilever within reasonable limits an upward load of 10 kN is applied at mid-span. Construct the S.F. and B.M. diagrams for the cantilever and hence determine the values of the reaction force and moment at the support. [30 kN, 70 kN m.]

3.8 (A). A beam 4.2 m long overhangs each of two simple supports by 0.6 m. The beam carries a uniformly distributed load of 30 kN/m between supports together with concentrated loads of 20 kN and 30 kN at the two ends. Sketch the S.F. and B.M. diagrams for the beam and hence determine the position of any points of contraflexure. [S.F. −20, +43, −47, +30 kN; B.M. −12, 18.75, −18 kN m; 0.313 and 2.553 m from l.h. support.]

3.9 (A/B). A beam $ABCDE$, with A on the left, is 7 m long and is simply supported at B and E. The lengths of the various portions are $AB = 1{\cdot}5$ m, $BC = 1{\cdot}5$ m, $CD = 1$ m and $DE = 3$ m. There is a uniformly distributed load of 15 kN/m between B and a point 2 m to the right of B and concentrated loads of 20 kN act at A and D with one of 50 kN at C.

(a) Draw the S.F. diagrams and hence determine the position from A at which the S.F. is zero.
(b) Determine the value of the B.M. at this point.
(c) Sketch the B.M. diagram approximately to scale, quoting the principal values.

[3.32 m; 69.8 kN m; 0, −30, 69.1, 68.1, 0 kN m.]

3.10 (A/B). A beam $ABCDE$ is simply supported at A and D. It carries the following loading: a distributed load of 30 kN/m between A and B; a concentrated load of 20 kN at B; a concentrated load of 20 kN at C; a concentrated load of 10 kN at E; a distributed load of 60 kN/m between D and E. Span $AB = 1.5$ m, $BC = CD = DE = 1$ m. Calculate the value of the reactions at A and D and hence draw the S.F. and B.M. diagrams. What are the magnitude and position of the maximum B.M. on the beam? [41.1, 113.9 kN; 28.15 kN m; 1.37 m from A.]

3.11 (B). A beam, 12 m long, is to be simply supported at 2 m from each end and to carry a u.d.l. of 30 kN/m together with a 30 kN point load at the right-hand end. For ease of transportation the beam is to be jointed in two places, one joint being situated 5 m from the left-hand end. What load (to the nearest kN) must be applied to the left-hand end to ensure that there is no B.M. at the joint (i.e. the joint is to be a point of contraflexure)? What will then be the best position on the beam for the other joint? Determine the position and magnitude of the maximum B.M. present on the beam. [114 kN, 1.6 m from r.h. reaction; 4.7 m from l.h. reaction; 43.35 kN m.]

3.12 (B). A horizontal beam AB is 4 m long and of constant flexural rigidity. It is rigidly built-in at the left-hand end A and simply supported on a non-yielding support at the right-hand end B. The beam carries uniformly distributed vertical loading of 18 kN/m over its whole length, together with a vertical downward load of 10 kN at 2.5 m from the end A. Sketch the S.F. and B.M. diagrams for the beam, indicating all main values.
[I. Struct. E.] [S.F. 45, −10, −37.6 kN; B.M. −18.6, +36.15 kN m.]

3.13 (B). A beam ABC, 6 m long, is simply-supported at the left-hand end A and at B 1 m from the right-hand end C. The beam is of weight 100 N/metre run.

(a) Determine the reactions at A and B.
(b) Construct to scales of 20 mm = 1 m and 20 mm = 100 N, the shearing-force diagram for the beam, indicating thereon the principal values.
(c) Determine the magnitude and position of the maximum bending moment. (You may, if you so wish, deduce the answers from the shearing force diagram without constructing a full or partial bending-moment diagram.)

[C.G.] [240 N, 360 N, 288 Nm, 2.4 m from A.]

3.14 (B). A beam $ABCD$, 6 m long, is simply-supported at the right-hand end D and at a point B 1m from the left-hand end A. It carries a vertical load of 10 kN at A, a second concentrated load of 20 kN at C, 3 m from D, and a uniformly distributed load of 10 kN/m between C and D. Determine:

(a) the values of the reactions at B and D,
(b) the position and magnitude of the maximum bending moment.

[33 kN, 27 kN, 2.7 m from D, 36.45 k Nm.]

3.15 (B). A beam $ABCD$ is simply supported at B and C with $AB = CD = 2$ m; $BC = 4$ m. It carries a point load of 60 kN at the free end A, a uniformly distributed load of 60 kN/m between B and C and an anticlockwise moment of

80 kN m in the plane of the beam applied at the free end D. Sketch and dimension the S.F. and B.M. diagrams, and determine the position and magnitude of the maximum bending moment.

[E.I.E.] [S.F. -60, $+170$, -70 kN; B.M. -120, $+120.1$, $+80$ kN m; 120.1 kN m at 2.83 m to right of B.]

3.16 (B). A beam $ABCDE$ is 4.6 m in length and loaded as shown in Fig. 3.19. Draw the S.F. and B.M. diagrams for the beam, indicating all major values.

[I.E.I.] [S.F. 28.27, 7.06, -12.94, -30.94, $+18$, 0; B.M. 28.27, 7.06, 15.53, -10.8.]

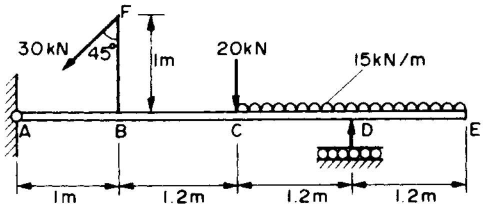


Fig. 3.19.

3.17 (B). A simply supported beam has a span of 6 m and carries a distributed load which varies in a linear manner from 30 kN/m at one support to 90 kN/m at the other support. Locate the point of maximum bending moment and calculate the value of this maximum. Sketch the S.F. and B.M. diagrams.

[U.L.] [3.25 m from l.h. end; 272 kN m.]

3.18 (B). Obtain the relationship between the bending moment, shearing force, and intensity of loading of a laterally loaded beam. A simply supported beam of span L carries a distributed load of intensity kx^2/L^2, where x is measured from one support towards the other. Determine: (a) the location and magnitude of the greatest bending moment, (b) the support reactions. [U. Birm.] [0.0394 kL^2 at 0.63 of span; $kL/12$ and $kL/4$.]

3.19 (B). A beam ABC is continuous over two spans. It is built-in at A, supported on rollers at B and C and contains a hinge at the centre of the span AB. The loading consists of a uniformly distributed load of total weight 20 kN on the 7 m span AB and a concentrated load of 30 kN at the centre of the 3 m span BC. Sketch the S.F. and B.M. diagrams, indicating the magnitudes of all important values.

[I.E.I.] [S.F. 5, -15, 26.67, -3.33 kN; B.M. 4.38, -35, $+5$ kN m.]

3.20 (B). A log of wood 225 mm square cross-section and 5 m in length is rendered impervious to water and floats in a horizontal position in fresh water. It is loaded at the centre with a load just sufficient to sink it completely. Draw S.F. and B.M. diagrams for the condition when this load is applied, stating their maximum values. Take the density of wood as 770 kg/m^3 and of water as 1000 kg/m^3. [S.F. 0, ± 0.285, 0 kN; B.M. 0, 0.356, 0 kN m.]

3.21 (B). A simply supported beam is 3 m long and carries a vertical load of 5 kN at a point 1 m from the left-hand end. At a section 2 m from the left-hand end a clockwise couple of 3 kN m is exerted, the axis of the couple being horizontal and perpendicular to the longitudinal axis of the beam. Draw to scale the B.M. and S.F. diagrams and mark on them the principal dimensions. [I.Mech.E.] [S.F. 2.33, -2.67 kN; B.M. 2.33, -0.34, $+2.67$ kN m.]