CHAPTER 2

STRUTS

Summary

The allowable stresses and end loads given by Euler's theory for struts with varying end conditions are given in Table 2.1.

Table 2.1.

End condition	Fixed-free	Pinned–pinned (or rounded)	Fixed–pinned	Fixed–fixed
Euler load P_e	$\dfrac{\pi^2 EI}{4L^2}$	$\dfrac{\pi^2 EI}{L^2}$	$\dfrac{2\pi^2 EI}{L^2}$	$\dfrac{4\pi^2 EI}{L^2}$
	or, writing $I = Ak^2$, where k = radius of gyration			
	$\dfrac{\pi^2 EA}{4(L/k)^2}$	$\dfrac{\pi^2 EA}{(L/k)^2}$	$\dfrac{2\pi^2 EA}{(L/k)^2}$	$\dfrac{4\pi^2 EA}{(L/k)^2}$
Euler stress σ_e	$\dfrac{\pi^2 E}{4(L/k)^2}$	$\dfrac{\pi^2 E}{(L/k)^2}$	$\dfrac{2\pi^2 E}{(L/k)^2}$	$\dfrac{4\pi^2 E}{(L/k)^2}$

Here L is the length of the strut and the term L/k is known as the *slenderness ratio.*

Validity limit for Euler formulae

$$L/k = \sqrt{\left(\frac{C\pi^2 E}{\sigma_y}\right)}$$

where C is a constant depending on the end condition of the strut.

Rankine–Gordon Formula

$$\sigma = \frac{\sigma_y}{1 + a(L/k)^2}$$

where $a = (\sigma_y/\pi^2 E)$ theoretically but is usually found by experiment. Typical values are given in Table 2.2.

Table 2.2.

Material	Compressive yield stress (MN/m^2)	a	
		Pinned ends	Fixed ends
Mild steel	315	1/7500	1/30 000
Cast iron	540	1/1600	1/64 000
Timber	35	1/3000	1/12 000

N.B. The value of a for pinned ends is always four times that for fixed ends

Perry–Robertson Formula

$$N\sigma = \frac{[\sigma_y + (\eta+1)\sigma_e]}{2} - \sqrt{\left\{\left[\frac{\sigma_y + (\eta+1)\sigma_e}{2}\right]^2 - \sigma_y\sigma_e\right\}}$$

where η is a constant depending on the material.

For a brittle material

$$\eta = 0.015L/k$$

For a ductile material

$$\eta = 0.3\left(\frac{L}{100k}\right)^2$$

These values will be modified for eccentric loading conditions. The Perry–Robertson formula is the basis of BS 449 as shown in §2.7.

Struts with initial curvature

$$\text{Maximum deflection } \delta_{\max} = \left[\frac{P_e}{(P_e - P)}\right] C_0$$

$$\text{Maximum stress } \sigma_{\max} = \frac{P}{A} \pm \left[\frac{PP_e}{(P_e - P)}\right]\frac{C_0 h}{I}$$

where C_0 is the initial central deflection and h is the distance of the highest strained fibre from the neutral axis (N.A.).

Smith–Southwell formula for eccentrically loaded struts

With pinned ends the maximum stress reached in the strut is given by

$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{L}{2}\sqrt{\left(\frac{\sigma}{Ek^2}\right)}\right]$$

or

$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{1}{2}\pi\sqrt{\left(\frac{\sigma}{\sigma_E}\right)}\right]$$

where e is the eccentricity of loading, h is the distance of the highest strained fibre from the N.A., k is the minimum radius of gyration of the cross-section, and σ is the applied load/cross-sectional area.

Since the required allowable stress σ cannot be obtained directly from this equation a solution is obtained graphically or by trial and error.

With other end conditions the value L in the above formula should be replaced by the appropriate *equivalent strut* length (see §2.2).

Webb's approximation for the Smith–Southwell formula

$$\sigma_{\max} = \frac{P}{A}\left[1 + \frac{eh}{k^2}\left(\frac{P_e + 0.26P}{P_e - P}\right)\right]$$

Laterally loaded struts

(a) Central concentrated load

$$\text{Maximum deflection} = \frac{W}{2nP}\left[\tan\frac{nL}{2} - \frac{nL}{2}\right]$$

$$\text{maximum bending moment (B.M.)} = \frac{W}{2n}\tan\frac{nL}{2}$$

(b) Uniformly distributed load

$$\text{Maximum deflection} = \frac{w}{n^2P}\left[\left(\sec\frac{nL}{2} - 1\right) - \frac{n^2L^2}{8}\right]$$

$$\text{maximum B.M.} = \frac{w}{n^2}\left(\sec\frac{nL}{2} - 1\right)$$

Introduction

Structural members which carry compressive loads may be divided into two broad categories depending on their relative lengths and cross-sectional dimensions. Short, thick members are generally termed *columns* and these usually fail by *crushing* when the yield stress of the material in compression is exceeded. Long, slender columns or *struts*, however, fail by *buckling* some time before the yield stress in compression is reached. The buckling occurs owing to one or more of the following reasons:

(a) the strut may not be perfectly straight initially;
(b) the load may not be applied exactly along the axis of the strut;
(c) one part of the material may yield in compression more readily than others owing to some lack of uniformity in the material properties throughout the strut.

At values of load below the buckling load a strut will be in stable eqilibrium where the displacement caused by any lateral disturbance will be totally recovered when the disturbance is removed. At the buckling load the strut is said to be in a state of neutral equilibrium, and theoretically it should then be possible to gently deflect the strut into a simple sine wave provided that the amplitude of the wave is kept small. This can be demonstrated quite simply using long thin strips of metal, e.g. a metal rule, and gentle application of compressive loads.

Theoretically, it is possible for struts to achieve a condition of unstable equilibrium with loads exceeding the buckling load, any slight lateral disturbance then causing failure by buckling; this condition is never achieved in practice under static load conditions. Buckling occurs immediately at the point where the buckling load is reached owing to the reasons stated earlier.

The above comments and the contents of this chapter refer to the *elastic* stability of struts only. It must also be remembered that struts can also fail plastically, and in this case the failure is irreversible.

2.1. Euler's theory

(a) Strut with pinned ends

Consider the axially loaded strut shown in Fig. 2.1 subjected to the crippling load P_e producing a deflection y at a distance x from one end. Assume that the ends are either pin-jointed or rounded so that there is no moment at either end.

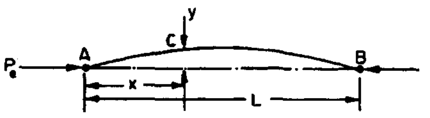

Fig. 2.1. Strut with axial load and pinned ends.

$$\text{B.M. at } C = EI\frac{d^2y}{dx^2} = -P_e y$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} + P_e y = 0$$

$$\therefore \qquad \frac{d^2y}{dx^2} + \frac{P_e}{EI}y = 0$$

i.e. in operator form, with $D \equiv d/dx$,

$$(D^2 + n^2)y = 0, \quad \text{where } n^2 = P_e/EI$$

This is a second-order differential equation which has a solution of the form

$$y = A\cos nx + B\sin nx$$

$$\text{i.e.} \qquad y = A\cos\sqrt{\left(\frac{P_e}{EI}\right)}x + B\sin\sqrt{\left(\frac{P_e}{EI}\right)}x$$

Now at $x = 0$, $y = 0$ $\qquad \therefore A = 0$

and at $x = L$, $y = 0$ $\qquad \therefore B\sin L\sqrt{(P_e/EI)} = 0$

$$\therefore \qquad \text{either } B = 0 \text{ or } \sin L\sqrt{\left(\frac{P_e}{EI}\right)} = 0$$

If $B = 0$ then $y = 0$ and the strut has not yet buckled. Thus the solution required is

$$\sin L\sqrt{\left(\frac{P_e}{EI}\right)} = 0 \qquad \therefore L\sqrt{\left(\frac{P_e}{EI}\right)} = \pi$$

$$\therefore \qquad P_e = \frac{\pi^2 EI}{L^2} \tag{2.1}$$

It should be noted that other solutions exist for the equation

$$\sin L\sqrt{\left(\frac{P}{EI}\right)} = 0 \quad \text{i.e.} \quad \sin nL = 0$$

The solution chosen of $nL = \pi$ is just one particular solution; the solutions $nL = 2\pi$, 3π, 5π, etc., are equally as valid mathematically and they do, in fact, produce values of P_e which are equally valid for modes of buckling of the strut different from that of the simple bow of Fig. 2.1. Theoretically, therefore, there are an infinite number of values of P_e, each corresponding with a different mode of buckling. The value selected above is the so-called *fundamental* mode value and is the lowest critical load producing the single-bow buckling condition. The solution $nL = 2\pi$ produces buckling in two half-waves, 3π in three half-waves, etc., as shown in Fig. 2.2. If load is applied sufficiently quickly to the strut, it is possible to pass through the fundamental mode and to achieve at least one of the other modes which are theoretically possible. In practical loading situations, however, this is rarely achieved since the high stress associated with the first critical condition generally ensures immediate collapse. The buckling load of a strut with pinned ends is, therefore, for all practical purposes, given by eqn. (2.1).

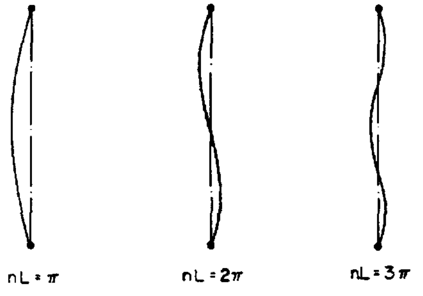

Fig. 2.2. Strut failure modes.

(b) One end fixed, the other free

Consider now the strut of Fig. 2.3 with the origin at the fixed end.

$$\text{B.M. at } C = EI\frac{d^2y}{dx^2} = +P(a-y)$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{Pa}{EI}$$

$$\therefore \quad (D^2 + n^2)y = n^2a \tag{2.2}$$

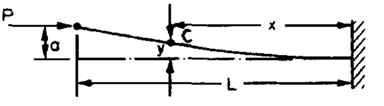

Fig. 2.3. Fixed–free strut.

N.B.–It is always convenient to arrange the diagram and origin such that the differential equation is achieved in the above form since the solution will then always be of the form

$$y = A\cos nx + B\sin nx + \text{(particular solution)}$$

The *particular solution* is a particular value of y which satisfies eqn. (2.2), and in this case can be shown to be $y = a$.

$$\therefore \qquad y = A\cos nx + B\sin nx + a$$

Now when $x = 0$, $y = 0$

$$\therefore \qquad A = -a$$

when $x = 0$, $dy/dx = 0$

$$\therefore \qquad B = 0$$

$$\therefore \qquad y = -a\cos nx + a$$

But when $x = L$, $y = a$

$$\therefore \qquad a = -a\cos nL + a$$

$$0 = \cos nL$$

The fundamental mode of buckling in this case therefore is given when $nL = \frac{1}{2}\pi$.

$$\therefore \qquad L\sqrt{\left(\frac{P}{EI}\right)} = \frac{\pi}{2}$$

or

$$\boldsymbol{P_e = \frac{\pi^2 EI}{4L^2}} \qquad (2.3)$$

(c) Fixed ends

Consider the strut of Fig. 2.4 *with the origin at the centre.*

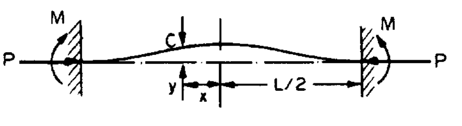

Fig. 2.4. Strut with fixed ends.

In this case the B.M. at C is given by

$$EI\frac{d^2y}{dx^2} = M - Py$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{M}{EI}$$

$$(D^2 + n^2)y = M/EI$$

Here the particular solution is

$$y = \frac{M}{n^2EI} = \frac{M}{P}$$

$$\therefore \qquad y = A\cos nx + B\sin nx + M/P$$

Now when $x = 0$, $dy/dx = 0 \therefore B = 0$

and when $x = \frac{1}{2}L$, $y = 0$ $\quad \therefore A = \dfrac{-M}{P}\sec\dfrac{nL}{2}$

$$\therefore \qquad y = -\frac{M}{P}\sec\frac{nL}{2}\cos nx + \frac{M}{P}$$

But when $x = \frac{1}{2}L$, dy/dx is also zero,

$$\therefore \qquad 0 = \frac{nM}{P}\sec\frac{nL}{2}\sin\frac{nL}{2}$$

$$0 = \frac{nM}{P}\tan\frac{nL}{2}$$

The fundamental buckling mode is then given when $nL/2 = \pi$

$$\therefore \qquad \frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)} = \pi$$

or

$$\mathbf{P_e = \frac{4\pi^2 EI}{L^2}} \tag{2.4}$$

(d) One end fixed, the other pinned

In order to maintain the pin-joint on the horizontal axis of the unloaded strut, it is necessary in this case to introduce a vertical load F at the pin (Fig. 2.5). The moment of F about the built-in end then balances the fixing moment.

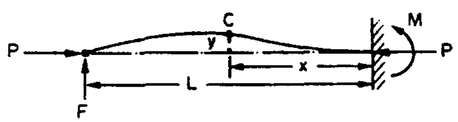

Fig. 2.5. Strut with one end pinned, the other fixed.

With the origin at the built-in end the B.M. at C is

$$EI\frac{d^2y}{dx^2} = -Py + F(L - x)$$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}y = \frac{F}{EI}(L - x)$$

$$(D^2 + n^2)y = \frac{F}{EI}(L - x)$$

The particular solution is

$$y = \frac{F}{n^2EI}(L - x) = \frac{F}{P}(L - x)$$

The full solution is therefore

$$y = A\cos nx + B\sin nx + \frac{F}{P}(L - x)$$

When $x = 0$, $y = 0$, $\quad \therefore A = -\dfrac{FL}{P}$

When $x = 0$, $dy/dx = 0$, $\therefore B = \dfrac{F}{nP}$

$$y = -\frac{FL}{P}\cos nx + \frac{F}{nP}\sin nx + \frac{F}{P}(L - x)$$

$$= \frac{F}{nP}[-nL\cos nx + \sin nx + n(L - x)]$$

But when $x = L$, $y = 0$

$\therefore$

$$nL\cos nL = \sin nL$$

$$\tan nL = nL$$

The lowest value of nL (neglecting zero) which satisfies this condition and which therefore produces the fundamental buckling condition is $nL = 4.5$ radians.

$\therefore$

$$L\sqrt{\left(\frac{P}{EI}\right)} = 4.5$$

or

$$P_e = \frac{20.25EI}{L^2} \qquad (2.5)$$

or, approximately

$$\boldsymbol{P_e = \frac{2\pi^2EI}{L^2}} \qquad (2.6)$$

2.2. Equivalent strut length

Having derived the result for the buckling load of a strut with pinned ends the Euler loads for other end conditions may all be written in the same form,

i.e.

$$\boldsymbol{P_e = \frac{\pi^2EI}{l^2}} \qquad (2.7)$$

where l is the *equivalent length* of the strut and can be related to the actual length of the strut depending on the end conditions. The equivalent length is found to be the length of a simple bow (half sine-wave) in each of the strut deflection curves shown in Fig. 2.6. The buckling load for each end condition shown is then readily obtained.

The use of the equivalent length is not restricted to the Euler theory and it will be used in other derivations later.

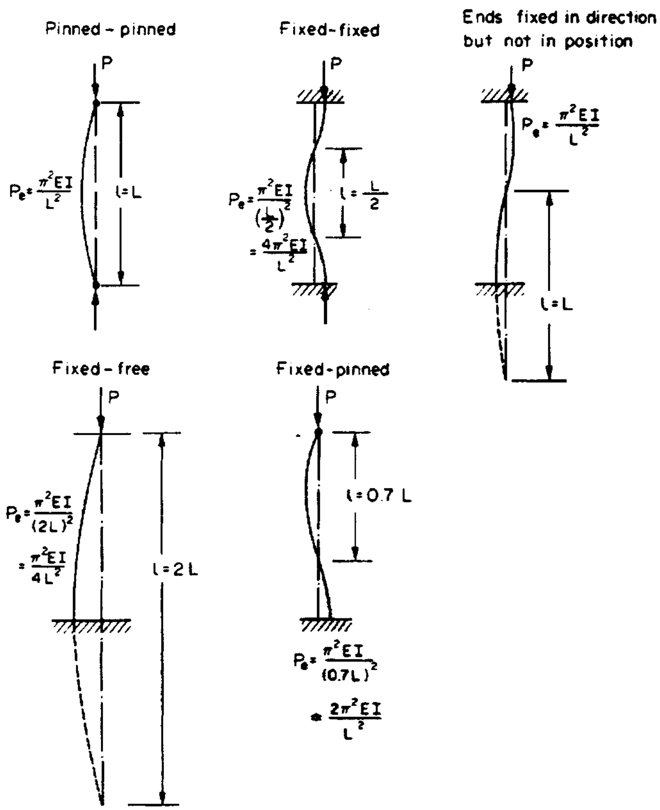

Fig. 2.6. "Equivalent length" of struts with different end conditions. In each case l is the length of a single bow.

2.3. Comparison of Euler theory with experimental results (see Fig. 2.7)

Between $L/k = 40$ and $L/k = 100$ neither the Euler results nor the yield stress are close to the experimental values, each suggesting a critical load which is in excess of that which is actually required for failure–a very unsafe situation! Other formulae have therefore been derived to attempt to obtain closer agreement between the actual failing load and the predicted value in this particular range of slenderness ratio.

(a) Straight-line formula

$$P = \sigma_y A[1 - n(L/k)] \tag{2.8}$$

the value of n depending on the material used and the end condition.

(b) Johnson parabolic formula

$$P = \sigma_y A[1 - b(L/k)^2] \tag{2.9}$$

the value of b depending also on the end condition.

Neither of the above formulae proved to be very successful, and they were replaced by:

(c) Rankine–Gordon formula

$$\frac{1}{P_R} = \frac{1}{P_e} + \frac{1}{P_c} \tag{2.10}$$

where P_e is the Euler buckling load and P_c is the crushing (compressive yield) load $= \sigma_y A$.
This formula has been widely used and is discussed fully in §2.5.

2.4. Euler "validity limit"

From the graph of Fig. 2.7 and the comments above, it is evident that the Euler theory is unsafe for small L/k ratios. It is useful, therefore, to determine the limiting value of L/k below which the Euler theory should not be applied; this is termed the *validity limit*.

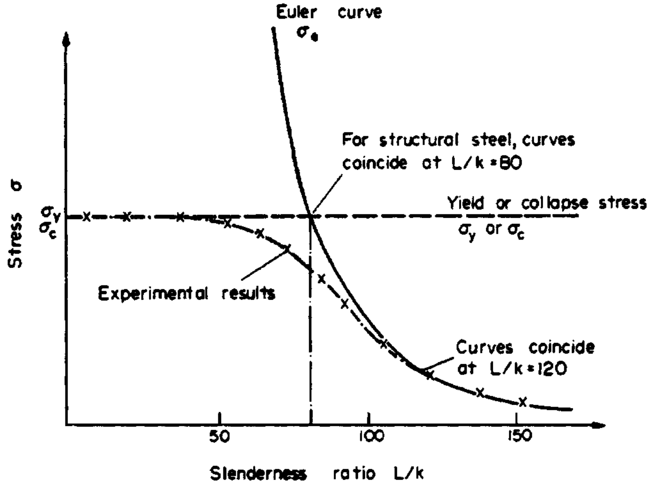

Fig. 2.7. Comparison of experimental results with Euler curve.

The validity limit is taken to be the point where the Euler σ_e equals the yield or crushing stress σ_y, i.e. the point where the strut load

$$P = \sigma_y A$$

Now the Euler load can be written in the form

$$P_e = C\frac{\pi^2 EI}{L^2} = C\frac{\pi^2 EAk^2}{L^2}$$

where C is a constant depending on the end condition of the strut.
Therefore in the limiting condition

$$\sigma_y A = C\frac{\pi^2 EAk^2}{L^2}$$

$$\frac{L}{k}=\sqrt{\left(\frac{C\pi^2E}{\sigma_y}\right)}$$

The value of this expression will vary with the type of end condition; as an example, low carbon steel struts with pinned ends give $L/k \simeq 80$.

2.5. Rankine or Rankine–Gordon formula

As stated above, the Rankine formula is a combination of the Euler and crushing loads for a strut

$$\frac{1}{P_R}=\frac{1}{P_e}+\frac{1}{P_c}$$

For very short struts P_e is very large; $1/P_e$ can therefore be neglected and $P_R = P_c$. For very long struts P_e is very small and $1/P_e$ is very large so that $1/P_c$ can be neglected. Thus $P_R = P_e$.

The Rankine formula is therefore valid for extreme values of L/k. It is also found to be fairly accurate for the intermediate values in the range under consideration. Thus, re-writing the formula in terms of stresses,

$$\frac{1}{\sigma A}=\frac{1}{\sigma_e A}+\frac{1}{\sigma_y A}$$

i.e.

$$\frac{1}{\sigma}=\frac{1}{\sigma_e}+\frac{1}{\sigma_y}=\frac{\sigma_e+\sigma_y}{\sigma_e\sigma_y}$$

$$\sigma=\frac{\sigma_e\sigma_y}{\sigma_e+\sigma_y}=\frac{\sigma_y}{[1+(\sigma_y/\sigma_e)]}$$

For a strut with both ends pinned

$$\sigma_e=\frac{\pi^2E}{(L/k)^2}$$

$\therefore$

$$\sigma=\frac{\sigma_y}{1+\dfrac{\sigma_y}{\pi^2E}\left(\dfrac{L}{k}\right)^2}$$

i.e. Rankine stress

$$\boldsymbol{\sigma_R=\frac{\sigma_y}{1+a(L/k)^2}} \tag{2.11}$$

where $a = \sigma_y/\pi^2E$, theoretically, but having a value normally found by experiment for various materials. This will take into account other types of end condition.

Therefore Rankine load

$$\boldsymbol{P_R=\frac{\sigma_y A}{1+a(L/k)^2}} \tag{2.12}$$

Typical values of a for use in the Rankine formula are given in Table 2.3.

However, since the values of a are not exactly equal to the theoretical values, the Rankine loads for long struts will not be identical to those estimated by the Euler theory as suggested earlier.

Table 2.3.

Material	σ_y or σ_c (MN/m^2)	a	
		Pinned ends	Fixed ends
Low carbon steel	315	1/7500	1/30 000
Cast iron	540	1/1600	1/64 000
Timber	35	1/3000	1/12 000

N.B. a for pinned ends $= 4 \times$ (a for fixed ends)

2.6. Perry–Robertson formula

The Perry–Robertson proof is based on the assumption that any imperfections in the strut, through faulty workmanship or material or eccentricity of loading, can be allowed for by giving the strut an initial curvature. For ease of calculation this is assumed to be a cosine curve, although the actual shape assumed has very little effect on the result.

Consider, therefore, the strut AB of Fig. 2.8, of length L and pin-jointed at the ends. The initial curvature y_0 at any distance x from the centre is then given by

$$y_0 = C_0 \cos \frac{\pi x}{L}$$

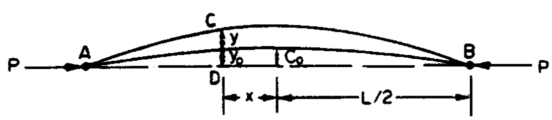

Fig. 2.8. Strut with initial curvature.

If a load P is now applied at the ends, this deflection will be increased to $y + y_0$.

$$\therefore \qquad BM_c = EI\frac{d^2y}{dx^2} = -P\left(y + C_0 \cos\frac{\pi x}{L}\right)$$

$$\therefore \qquad \frac{d^2y}{dx^2} + \frac{P}{EI}\left(y + C_0\cos\frac{\pi x}{L}\right) = 0$$

the solution of which is

$$y = A\sin\sqrt{\left(\frac{P}{EI}\right)}x + B\cos\sqrt{\left(\frac{P}{EI}\right)}x + \left[\left(\frac{PC_0}{EI}\cos\frac{\pi x}{L}\right) \Big/ \left(\frac{\pi^2}{L^2} - \frac{P}{EI}\right)\right]$$

where A and B are the constants of integration.
Now when $x = \pm L/2$, $y = 0$

$$\therefore \qquad A = B = 0$$

$$\therefore \qquad y = \left[\left(\frac{PC_0}{EI}\cos\frac{\pi x}{L}\right) \Big/ \left(\frac{\pi^2}{L^2} - \frac{P}{EI}\right)\right] = \left[\left(PC_0\cos\frac{\pi x}{L}\right) \Big/ \left(\frac{\pi^2 EI}{L^2} - P\right)\right]$$

Therefore dividing through, top and bottom, by A,

$$y = \left[\left(\frac{P}{A} C_0 \cos\frac{\pi x}{L}\right) \Big/ \left(\frac{\pi^2 EI}{L^2 A} - \frac{P}{A}\right)\right]$$

But $P/A = \sigma$ and $(\pi^2 EI)/(L^2 A) = \sigma_e$ (the Euler stress for pin-ended struts)

$$\therefore \qquad y = \frac{\sigma}{(\sigma_e - \sigma)} C_0 \cos\frac{\pi x}{L}$$

Therefore total deflection at any point is given by

$$y + y_0 = \left[\frac{\sigma}{(\sigma_e - \sigma)}\right] C_0 \cos\frac{\pi x}{L} + C_0 \cos\frac{\pi x}{L}$$

$$= \left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 \cos\frac{\pi x}{L} \qquad (2.13)$$

$$\therefore \qquad \text{Maximum deflection (when } x = 0) = \left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 \qquad (2.14)$$

$$\therefore \qquad \text{maximum B.M.} = P\left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 \qquad (2.15)$$

$$\therefore \qquad \text{maximum stress owing to bending} = \frac{My}{I} = \frac{P}{I}\left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 h$$

where h is the distance of the outside fibre from the N.A. of the strut.

Therefore the maximum stress owing to combined bending and thrust is given by

$$\sigma_{\max} = \frac{P}{I}\left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 h + \frac{P}{A} \qquad (2.16)$$

$$= \frac{P}{Ak^2}\left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 h + \frac{P}{A}$$

$$= \sigma\left[\frac{\eta\sigma_e}{(\sigma_e - \sigma)} + 1\right] \quad \text{where } \eta = \frac{C_0 h}{k^2}$$

If $\sigma_{\max} = \sigma_y$, the compressive yield stress for the material of the strut, the above equation when solved for σ gives

$$\sigma = \frac{[\sigma_y + (\eta + 1)\sigma_e]}{2} - \sqrt{\left\{\left[\frac{\sigma_y + (\eta + 1)\sigma_e}{2}\right]^2 - \sigma_y\sigma_e\right\}} \qquad (2.17)$$

This is the Perry–Robertson formula required. If the material is brittle, however, and failure is likely to occur in tension, then the sign between the two square-bracketed terms becomes positive and σ_y is the *tensile* yield strength.

2.7. British Standard procedure (BS 449)

With a *load factor N* applied, the Perry–Robertson equation becomes

$$N\sigma = \frac{[\sigma_y + (\eta + 1)\sigma_e]}{2} - \sqrt{\left\{\left[\frac{\sigma_y + (\eta + 1)\sigma_e}{2}\right]^2 - \sigma_y\sigma_e\right\}} \qquad (2.18)$$

With values for steel of $\sigma_y = 225$ MN/m^2, $E = 200$ GN/m^2, $N = 1.7$ and $\eta = 0.3(L/100k)^2$, the above equation gives the graph shown in Fig. 2.9. This graph then indicates the basis of design using BS449: 1959 (amended 1964). Allowable values are provided in the standard, however, in tabular form.

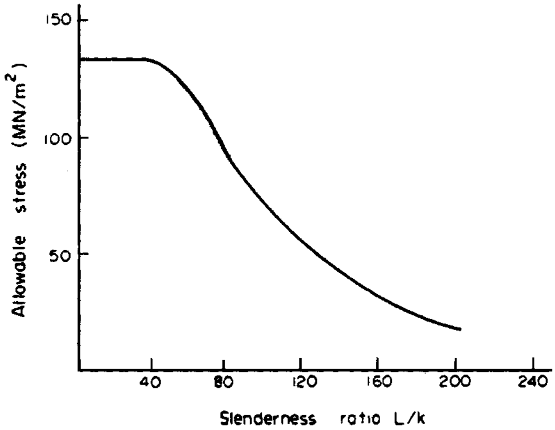

Fig. 2.9. Graph of allowable stress as given in BS 449: 1964 (in tabulated form) against slenderness ratio.

If, however, design is based on the *safety factor* method instead of the *load factor* method, then N is omitted and σ_y/n replaces σ_y in the formula, where n is the safety factor.

2.8. Struts with initial curvature

In §2.6 the Perry–Robertson equation was derived on the assumption that strut imperfections could be allowed for by giving the strut an initial curvature. This proof applies equally well, of course, for struts which have genuine initial curvatures and, provided the curvature is small, the precise shape of the curve has little effect on the end result.

Thus for an initial curvature with a central deflection C_0,

$$\text{maximum deflection} = \left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 = \left[\frac{P_e}{(P_e - P)}\right] C_0 \qquad (2.19)$$

$$\text{maximum B.M.} = P\left[\frac{\sigma_e}{(\sigma_e - \sigma)}\right] C_0 = \left[\frac{PP_e}{(P_e - P)}\right] C_0 \qquad (2.20)$$

and

$$\sigma_{\max} = \frac{P}{A} \pm \left[\frac{P\sigma_e}{(\sigma_e - \sigma)}\right] \frac{hC_0}{I}$$

$$= \frac{P}{A} \pm \left[\frac{PP_e}{(P_e - P)}\right]\frac{hC_0}{I} \tag{2.21}$$

where h is the distance from the N.A. to the outside fibres of the strut.

2.9. Struts with eccentric load

For eccentric loading at the ends of a strut Ayrton and Perry suggest that the Perry–Robertson formula can be modified by replacing C_0 by $(C_0 + 1.2e)$ where e is the eccentricity.
Then

$$\eta' = \eta + 1.2\frac{eh}{k^2} \tag{2.22}$$

and η' replaces η in the original Perry–Robertson equation.

(a) Pinned ends – the Smith–Southwell formula

For a more fundamental treatment consider the strut loaded as shown in Fig. 2.10 carrying a load P at an eccentricity e on one principal axis. In this case there is strictly no 'buckling" load as previously described since the strut will bend immediately load is applied, bending taking place about the other principal axis.

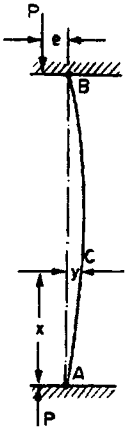

Fig. 2.10. Strut with eccentric load (pinned ends)

Applying a similar procedure to that used previously

$$\text{B.M. at } C = -P(y + e)$$

$$\therefore \quad EI\frac{d^2y}{dx^2} = -P(y + e)$$

$$\therefore \quad \frac{d^2y}{dx^2} + n^2(y + e) = 0$$

where $$n = \surd(P/EI)$$

This is a second-order differential equation, the solution of which is as follows:

$$y = A \sin nx + B \cos nx - e$$

Now when $x = 0$, $y = 0$

$$\therefore \quad B = e$$

and when $x = \frac{L}{2}, \frac{dy}{dx} = 0$

$$\therefore \quad 0 = nA \cos n\frac{L}{2} - ne \sin n\frac{L}{2}$$

$$\therefore \quad A = e \tan \frac{nL}{2}$$

$$\therefore \quad y + e = e \tan \frac{nL}{2} \sin nx + e \cos nx$$

$\therefore$ maximum deflection, when $x = L/2$ and $y = \delta$, is

$$\delta + e = e\frac{\sin^2 \frac{nL}{2}}{\cos \frac{nL}{2}} + e \cos \frac{nL}{2}$$

$$= e\frac{\left(\sin^2 \frac{nL}{2} + \cos^2 \frac{nL}{2}\right)}{\cos \frac{nL}{2}} = e \sec \frac{nL}{2} \qquad (2.23)$$

$$\therefore \quad \text{maximum B.M.} = P(\delta + e) = Pe \sec \frac{nL}{2} \qquad (2.24)$$

$$\therefore \quad \text{maximum stress owing to bending} = \frac{My}{I} = Pe \sec \frac{nL}{2} \times \frac{h}{I}$$

where h is the distance from the N.A. to the highest stressed fibre.

Therefore the total maximum compressive stress owing to combined bending and thrust, assuming a *ductile* material†, is given by

$$\sigma_{\max} = \frac{P}{A} + \left(Pe \sec \frac{nL}{2}\right)\frac{h}{I}$$

$$= \sigma \left[1 + \frac{eh}{k^2} \sec \frac{nL}{2}\right] \qquad (2.25)$$

$$= \sigma \left[1 + \frac{eh}{k^2} \sec \frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)}\right]$$

† For a brittle material which is relatively weak in tension it is the maximum tensile stress which becomes the criterion of failure and the bending and direct stress components are opposite in sign.

i.e.
$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{L}{2}\sqrt{\left(\frac{\sigma}{Ek^2}\right)}\right] \qquad (2.26)$$

This formula is known as the Smith–Southwell formula.

Unfortunately, since $\sigma = P/A$, the above equation represents a function of P (the required unknown) which can only be solved by trial and error or graphically. A good approximation however, is obtained as shown below:

Webb's approximation

From above
$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{nL}{2}\right] \qquad (2.25)\text{(bis)}$$

Let
$$\frac{nL}{2} = \theta$$

Then
$$\theta = \frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)} = \frac{\pi}{2}\sqrt{\left(\frac{L^2}{\pi^2}\frac{P}{EI}\right)} = \frac{\pi}{2}\sqrt{\left(\frac{P}{P_e}\right)}$$

Now for θ between 0 and $\pi/2$,

$$\sec\theta \simeq \frac{1 + 0.26\left(\dfrac{2\theta}{\pi}\right)^2}{1 - \left(\dfrac{2\theta}{\pi}\right)^2} = \frac{1 + 0.26\dfrac{P}{P_e}}{1 - \dfrac{P}{P_e}} = \frac{P_e + 0.26P}{P_e - P}$$

Therefore substituting in eqn. (2.25)

$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\left(\frac{P_e + 0.26P}{P_e - P}\right)\right]$$
$$= \frac{P}{A}\left[1 + \frac{eh}{k^2}\left(\frac{P_e + 0.26P}{P_e - P}\right)\right] \qquad (2.27)$$

where $\sigma_{\max}$ is the maximum allowable stress in the strut material, P_e is the Euler buckling load for axial loading, and P is the maximum allowable value of the eccentric load.

The above equation can be re-written into a more readily observed quadratic equation in P, thus:

$$P^2\left[1 - 0.26\frac{eh}{k^2}\right] - P\left[P_e\left(1 + \frac{eh}{k^2}\right) + \sigma_{\max}A\right] + \sigma_{\max}AP_e = 0 \qquad (2.28)$$

For any given eccentric load condition P is the only unknown and the equation can be readily solved.

(b) One end fixed, the other free

Consider the strut shown in Fig. 2.11.

$$BM_c = EI\frac{d^2y}{dx^2} = P(e_0 - y)$$

$$\therefore \qquad \frac{d^2y}{dx^2} + n^2 y = n^2 e_0$$

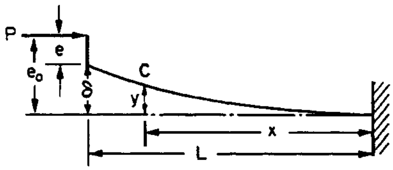

Fig. 2.11. Strut with eccentric load (one end fixed, the other free)

The solution of the expression is

$$y = A\cos nx + B\sin xn + e_0$$

At $x = 0$, $y = 0$ $\qquad \therefore A + e_0 = 0$ or $A = -e_0$

At $x = 0$, $dy/dx = 0$ $\qquad \therefore B = 0$

$$\therefore \qquad y = -e_0 \cos nx + e_0$$

Now at $x = L$, $y = \delta$

$$\begin{aligned} \therefore \qquad \delta &= -e_0 \cos nL + e_0 \\ &= e_0(1 - \cos nL) \\ &= (\delta + e)(1 - \cos nL) \\ &= \delta - \delta\cos nL + e - e\cos nL \end{aligned}$$

$$\therefore \qquad \delta\cos nL = e - e\cos nL$$

$$\therefore \qquad \delta = e(\sec nL - 1)$$

or

$$\delta + e = e\sec nL$$

This is the same form of solution as that obtained previously for pinned ends with L replaced by $2L$, i.e. the Smith–Southwell formula will apply in this case provided that the equivalent length of the strut ($l = 2L$) is used in place of L.

Thus the Smith–Southwell formula can be written in the form

$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{l}{2}\sqrt{\left(\frac{\sigma}{Ek^2}\right)}\right] \qquad (2.29)$$

the value of the equivalent length l to be used for any given end condition being given by the diagrams of Fig. 2.6, §2.2.

The exception to this rule, however, is the case of fixed ends where the only effect of eccentricity of loading is to increase the fixing moments within the supports at each end; there will be no effect on the deflection or stress in the strut itself. Thus, eccentricity of loading can be neglected in the case of fixed-ended struts – an important factor since most practical struts can be considered to be of this type.

2.10. Laterally loaded struts

(a) Central concentrated load

With the origin at the centre of the strut as shown in Fig. 2.12,

$$\text{B.M. at } C = EI\frac{d^2y}{dx^2} = -Py - \frac{W}{2}\left(\frac{L}{2} - x\right)$$

$$\frac{d^2y}{dx^2} + n^2y = -\frac{W}{2EI}\left(\frac{L}{2} - x\right)$$

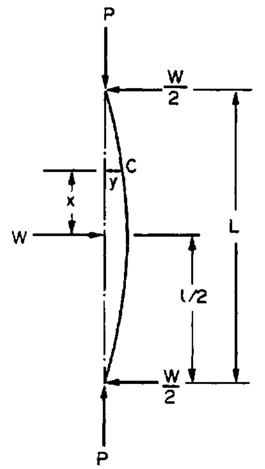

Fig. 2.12.

The solution of this equation is similar to that of §2.1(d),

i.e. $$y = A\cos nx + B\sin nx - \frac{W}{2P}\left(\frac{L}{2} - x\right)$$

Now when $x = 0$, $dy/dx = 0$ $\quad \therefore B = -\dfrac{W}{2nP}$

and when $x = L/2$, $y = 0$ $\quad \therefore A = \dfrac{W}{2nP}\tan\dfrac{nL}{2}$

$\therefore$ $$y = \frac{W}{2nP}\left[\tan\frac{nL}{2}\cos nx - \sin nx - n\left(\frac{L}{2} - x\right)\right]$$

The maximum deflection occurs where x is zero,

i.e. $$\boldsymbol{y_{\max} = \frac{W}{2nP}\left[\tan\frac{nL}{2} - \frac{nL}{2}\right]} \tag{2.30}$$

The maximum B.M. acting on the strut is at the same position and is given by

$$M_{\max} = -Py_{\max} - \frac{W}{2}\frac{L}{2}$$

$$= -\frac{W}{2n} \tan \frac{nL}{2} \tag{2.31}$$

(b) Uniformly distributed load

Consider now the uniformly loaded strut of Fig. 2.13 with the origin again selected at the centre but y measured from the maximum deflected position.

$$\text{B.M.}_{\cdot c} = EI\frac{d^2y}{dx^2} = P(\delta - y) + \frac{wL}{2}\left(\frac{L}{2} - x\right) - \frac{w}{2}\left(\frac{L}{2} - x\right)^2$$

$$= P\delta - Py + \frac{w}{2}\left(\frac{L^2}{4} - x^2\right)$$

$$\therefore \qquad \frac{d^2y}{dx^2} + n^2y = \frac{w}{2EI}\left(\frac{L^2}{4} - x^2\right) + n^2\delta$$

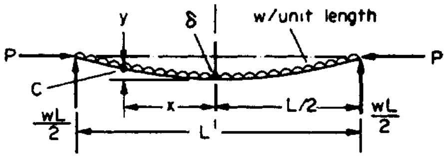

Fig. 2.13.

The solution of this equation is

$$y = A\cos nx + B\sin nx - \frac{w}{2P}\left(\frac{L^2}{4} - x^2\right) + \delta + \frac{2w}{2n^2P}$$

i.e.
$$y - \delta = A\cos nx + B\sin nx - \frac{w}{2P}\left(\frac{L^2}{4} - x^2 - \frac{2}{n^2}\right)$$

When $x = 0$, $dy/dx = 0 \quad \therefore B = 0$

When $x = L/2$, $y = \delta \quad \therefore A = \dfrac{w}{n^2P}\sec\dfrac{nL}{2}$

$$\therefore \qquad y - \delta = \frac{w}{n^2P}\left[\left(\sec\frac{nL}{2}\cos nx - 1\right) - n^2\left(\frac{L^2}{8} - \frac{x^2}{2}\right)\right]$$

Thus the maximum deflection δ, when $y = 0$ and $x = 0$, is given by

$$\delta = y_{max} = \frac{w}{n^2p}\left[\left(\sec\frac{nL}{2} - 1\right) - \frac{n^2L^2}{8}\right] \tag{2.32}$$

and the maximum B.M. is

$$M_{max} = P\delta + \frac{wL^2}{8} = \frac{w}{n^2}\left(\sec\frac{nL}{2} - 1\right) \tag{2.33}$$

In the case of a member carrying a tensile load (i.e. a *tie*) together with a uniformly distributed load, the above procedure applies with the sign for P reversed. The relevant differential expression then becomes

$$\frac{d^2y}{dx^2} - n^2y = \frac{w}{2EI}\left[\frac{L^2}{4} - x^2\right] + n^2\delta$$

i.e. $(D^2 - n^2)y$ in place of $(D^2 + n^2)y$ as usual.

The solution of this equation involves hyperbolic functions but remains of identical form to that obtained previously,

i.e. $$\mathbf{M = A\ \cosh nx + B\ \sinh nx + \text{etc.}}$$

giving $$M_{\max} = \frac{w}{n^2}\left(\operatorname{sech}\frac{nL}{2} - 1\right)$$

2.11. Alternative procedure for any strut-loading condition

If deflections are not the primary interest and only the B.M.'s and hence maximum stress are required, it is convenient to commence the analysis with a differential expression for the B.M. M.

This is most easily achieved by considering the moment divided into two parts:

(a) that due to the end load P;
(b) that due to any transverse load (M').

Thus total moment $M = -Py + M'$

Differentiating twice, $$\frac{d^2M}{dx^2} + P\frac{d^2y}{dx^2} = \frac{d^2M'}{dx^2}$$

But $$P\frac{d^2y}{dx^2} = \frac{P}{EI}\left(EI\frac{d^2y}{dx^2}\right) = n^2M$$

$\therefore$ $$\frac{d^2M}{dx^2} + n^2M = \frac{d^2M'}{dx^2}$$

The general solution will be of the form

$$M = A\cos nx + B\sin nx + \text{ particular solution}$$

Now for zero transverse load (or for any concentrated load) (d^2M'/dx^2) is zero, the particular solution is also zero, and the solution for the above expression is in the form

$$M = A\cos nx + B\sin nx$$

Thus, for an **eccentrically loaded strut** (Smith–Southwell):

$$\text{shear force} = \frac{dM}{dx} = 0 \qquad \text{when } x = 0 \qquad \therefore B = 0$$

and $$M = Pe \qquad \text{when } x = \tfrac{1}{2}L \quad \therefore A = Pe\sec\frac{nL}{2}$$

Therefore substituting, $M = Pe \sec \dfrac{nL}{2} \cos nx$

and $$\boldsymbol{M_{max} = Pe \sec \frac{nL}{2}} \quad \text{as before}$$

For a **central concentrated load** (see Fig. 2.12)

$$M' = \frac{W}{2}\left(\frac{L}{2} - x\right)$$

$$\therefore \qquad \frac{d^2M'}{dx^2} = 0 \quad \text{and the particular solution } = 0$$

$$\therefore \qquad M = A\cos nx + B\sin nx$$

$$\text{Shear force} = \frac{dM}{dx} = \frac{W}{2} \qquad \text{when } x = 0 \qquad \therefore B = \frac{W}{2n}$$

and $$M = 0 \qquad \text{when } x = \tfrac{1}{2}L \quad \therefore A = -\frac{W}{2n}\tan\frac{nL}{2}$$

$$\therefore \qquad M = -\frac{W}{2n}\left[\tan\frac{nL}{2}\cos nx + \sin nx\right]$$

and $$\boldsymbol{M_{max} = -\frac{W}{2n}\tan\frac{nL}{2}} \quad \text{as before}$$

For **a uniformly distributed lateral load** (see Fig. 2.13)

$$M' = \frac{w}{2}\left[\frac{L^2}{4} - x^2\right] \quad \text{(see page 47)}$$

$$\therefore \qquad \frac{d^2M'}{dx^2} = -w$$

Hence $$\frac{d^2M}{dx^2} + n^2M = -w \quad \text{and the particular integral is } \frac{w}{n^2}$$

$$\therefore \qquad M = A\cos nx + B\sin nx - w/n^2$$

Now when $x = 0$, $dM/dx = 0$ $\quad \therefore B = 0$

and when $x = L/2$, $M = 0$ $\quad \therefore A = \dfrac{w}{n^2}\sec\dfrac{nL}{2}$

$$\therefore \qquad M = \frac{w}{n^2}\left[\sec\frac{nL}{2}\cos nx - 1\right]$$

and $$\boldsymbol{M_{max} = \frac{w}{n^2}\left[\sec\frac{nL}{2} - 1\right]} \quad \text{as before}$$

2.12. Struts with unsymmetrical cross-sections

The formulae derived in the preceding paragraphs have assumed that buckling takes place about an axis of symmetry. Loading is then normally applied to produce bending on the strongest or major principal axis (that about which I has a maximum value) so that buckling is assumed to occur about the minor axis. It is also assumed that the end conditions allow rotation in this direction and this is normally achieved by loading through ball ends.

For sections with only one axis of symmetry, e.g. channel or T-sections, the shear centre is not coincident with the centroid and torsional effects are often introduced. These may, in some cases, affect the failure condition of the strut. Certainly, in the case of totally unsymmetrical sections, the loading condition always involves considerable torsion and the theoretical buckling load has little relevance. One popular form of section which falls in this category is the unequal-leg angle section.

Some sections, e.g. cruciform sections, are subject to both flexural and torsional buckling and the reader is referred to more advanced texts for the methods of treatment is such cases.

A special form of failure is associated with hollow low carbon steel columns with small thickness to diameter ratios when the strut is found to *crinkle*, i.e. the material forms into folds when the direct stress is approximately equal to the yield stress. Southwell has investigated this problem and produced the formula

$$\sigma = E\frac{t}{R}\left[\frac{1}{3(1-\nu^2)}\right]^{1/2}$$

where σ is the stress causing yielding, R is the mean radius of the column and t is the thickness. It should be noted, however, that this type of failure is not common since very small t/R ratios of the order of 1/400 are required before crinkling can occur.

Examples

Example 2.1

Two 300 mm × 120 mm I-section joists are united by 12 mm thick plates as shown in Fig. 2.14 to form a 7 m long stanchion. Given a factor of safety of 3, a compressive yield stress of 300 MN/m^2 and a constant a of 1/7500, determine the allowable load which can be carried by the stanchion according to the Rankine–Gordon formulae.

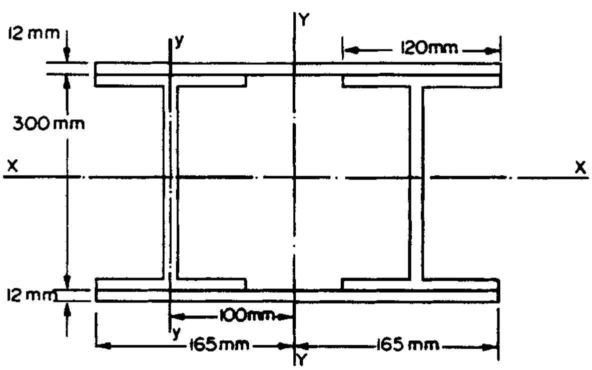

Fig. 2.14.

The relevant properties of each joist are:

$$I_{xx} = 96 \times 10^{-6}\ \text{m}^4, \quad I_{yy} = 4.2 \times 10^{-6}\ \text{m}^4, \quad A = 6 \times 10^{-3}\ \text{m}^2$$

Solution

For the strut of Fig. 2.14:

$$I_{xx} \text{ for joists} = 2 \times 96 \times 10^{-6} = 192 \times 10^{-6}\ \text{m}^4$$

$$I_{xx} \text{ for plates} = 0.33 \times \frac{0.324^3}{12} - \frac{0.33 \times 0.300^3}{12}$$

$$= \frac{0.33}{12}[0.034 - 0.027] = 192.5 \times 10^{-6}\ \text{m}^4$$

$$\therefore \qquad \text{total } I_{xx} = (192 + 192.5)10^{-6} = 384.5 \times 10^{-6}\ \text{m}^4$$

From the parallel axis theorem:

$$I_{yy} \text{ for joists} = 2(4.2 \times 10^{-6} + 6 \times 10^{-3} \times 0.1^2)$$

$$= 128.4 \times 10^{-6}\ \text{m}^4$$

and $$I_{yy} \text{ for plates} = 2 \times 0.012 \times \frac{0.33^3}{12} = 71.9 \times 10^{-6}\ \text{m}^4$$

$$\therefore \qquad \text{total } I_{yy} = 200.3 \times 10^{-6}\ \text{m}^4$$

Now the smallest value of the Rankine–Gordon stress σ_R is given when k, and hence I, is a minimum.

$$\therefore \qquad \text{smallest } I = I_{yy} = 200.3 \times 10^{-6} = Ak^2$$

$$\text{total area } A = 2 \times 6 \times 10^{-3} + 2 \times 0.33 \times 12 \times 10^{-3} = 19.92 \times 10^{-3}$$

$$\therefore \qquad 19.92 \times 10^{-3}k^2 = 200.3 \times 10^{-6}$$

$$\therefore \qquad k^2 = \frac{200.3 \times 10^{-6}}{19.92 \times 10^{-3}} = 10.05 \times 10^{-3}$$

$$\therefore \qquad \left(\frac{L}{k}\right)^2 = \frac{7^2}{10.05 \times 10^{-3}} = 4.9 \times 10^3$$

and $$\sigma_R = \frac{\sigma_y}{1 + a\left(\dfrac{L}{k}\right)^2} = \frac{300 \times 10^6}{1 + \dfrac{4.9 \times 10^3}{7500}}$$

$$\therefore \qquad = \frac{300 \times 10^6}{1.653} = 181.45\ \text{MN/m}^2$$

$$\therefore \qquad \text{allowable load} = \sigma_R \times A = 181.45 \times 10^6 \times 19.92 \times 10^{-3} = 3.61\ \text{MN}$$

With a factor of safety of 3 the maximum permissible load therefore becomes

$$P_{\max} = \frac{3.61 \times 10^6}{3} = \mathbf{1.203\ MN}$$

Example 2.2

An 8 m long column is constructed from two 400 mm × 250 mm I-section joists joined as shown in Fig. 2.15. One end of the column is arranged to be fixed and the other free and a load equal to one-third of the Euler load is applied. Determine the load factor provided if the Perry–Robertson formula is used as the basis for design.

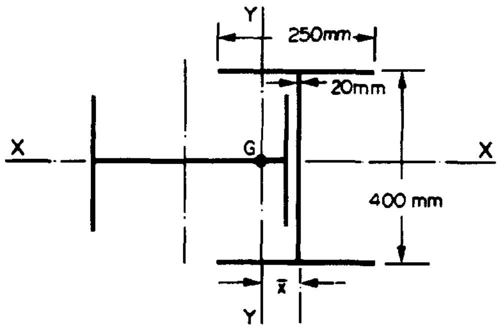

Fig. 2.15.

For each joist:

$$I_{max} = 213 \times 10^{-6}\ \text{m}^4, \quad I_{min} = 9.6 \times 10^{-6}\ \text{m}^4, \quad A = 8.4 \times 10^{-3}\ \text{m}^2,$$

with web and flange thicknesses of 20 mm. For the material of the joist, $E = 208\ \text{GN/m}^2$ and $\sigma_y = 270\ \text{MN/m}^2$.

Solution

To find the position of the centroid G of the built-up section take moments of area about the centre line of the vertical joist.

$$2 \times 8.4 \times 10^{-3}\,\bar{x} = 8.4 \times 10^{-3}(200 + 10)10^{-3}$$

$$\bar{x} = \frac{210}{2} \times 10^{-3} = 105\ \text{mm}$$

Now $$I_{xx} = (213 + 9.6)10^{-6} = 222.6 \times 10^{-6}\ \text{m}^4$$

and $$I_{yy} = [213 + 8.4(210 - 105)^2]10^{-6} + [9.6 + 8.4 \times 105^2]10^{-6}$$

i.e. greater than I_{xx}.

$\therefore$ $$\text{least } I = 222.6 \times 10^{-6}\ \text{m}^4$$

$\therefore$ $$\text{least } k^2 = \frac{222.6 \times 10^{-6}}{2 \times 8.4 \times 10^{-3}} = 13.25 \times 10^{-3}$$

Now Euler load for fixed–free ends

$$= \frac{\pi^2 EI}{4L^2} = \frac{\pi^2 \times 208 \times 10^9 \times 222.6 \times 10^{-6}}{4 \times 8^2}$$

$$= 1786 \times 10^3 = 1.79\ \text{MN}$$

Therefore actual load applied to the column

$$= \frac{1.79}{3} = 0.6 \text{ MN}$$

i.e.
$$\text{actual stress} = \frac{\text{load}}{\text{area}} = \frac{0.6 \times 10^6}{2 \times 8.4 \times 10^{-3}}$$
$$= \mathbf{35.7\ MN/m^2}$$

The Perry–Robertson constant is

$$\eta = 0.3\left(\frac{L}{100k}\right)^2 = 0.3\left(\frac{8^2}{10^4 \times 13.25 \times 10^{-3}}\right)$$
$$= 0.144$$

and
$$N\sigma = \frac{(\sigma_y + 1.144\sigma_e)}{2} - \sqrt{\left\{\left[\frac{(\sigma_y + 1.144\sigma_e)}{2}\right]^2 - \sigma_y\sigma_e\right\}}$$

But
$$\sigma_y = 270 \text{ MN/m}^2 \text{ and } \sigma_e = \frac{1.79 \times 10^6}{2 \times 8.4 \times 10^{-3}} = 106.5 \text{ MN/m}^2$$

i.e. in units of MN/m^2:

$$\therefore \qquad N\sigma = \frac{(270 + 121.8)}{2} - \sqrt{\left\{\left[\frac{270 + 121.8}{2}\right]^2 - 270 \times 106.5\right\}}$$
$$= 196 - 98 = 98$$

$$\therefore \qquad \text{load factor } N = \frac{98}{35.7} = \mathbf{2.75}$$

Example 2.3

Determine the maximum compressive stress set up in a 200 mm × 60 mm I-section girder carrying a load of 100 kN with an eccentricity of 6 mm from the critical axis of the section (see Fig. 2.16). Assume that the ends of the strut are pin-jointed and that the overall length is 4 m.

Take $I_{yy} = 3 \times 10^{-6}\ \text{m}^4$, $A = 6 \times 10^{-3}\ \text{m}^2$, $E = 207\ \text{GN/m}^2$.

Solution

Normal stress on the section

$$\sigma = \frac{P}{A} = \frac{100 \times 10^3}{6 \times 10^{-3}} = \frac{100}{6} \text{ MN/m}^2$$
$$I = Ak^2 = 3 \times 10^{-6} \text{ m}^4$$
$$\therefore \qquad k^2 = \frac{3 \times 10^{-6}}{6 \times 10^{-3}} = 5 \times 10^{-4} \text{ m}^2$$

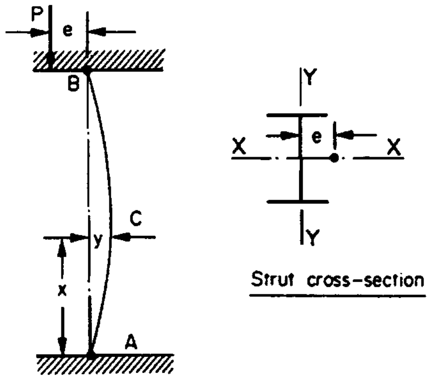

Fig. 2.16.

Now from eqn. (2.26)

$$\sigma_{\max} = \sigma\left[1 + \frac{eh}{k^2}\sec\frac{L}{2}\sqrt{\left(\frac{\sigma}{Ek^2}\right)}\right]$$

$\therefore$ with $e = 6$ mm and $h = 30$ mm

$$\sigma_{\max} = \frac{100}{6}\left[1 + \frac{30\times 6\times 10^{-6}}{5\times 10^{-4}}\sec 2\sqrt{\left(\frac{100\times 10^6\times 10^4}{6\times 207\times 10^9\times 5}\right)}\right]$$

$$= \frac{100}{6}[1 + 0.36\sec 2\sqrt{(0.161)}]$$

$$= \frac{100}{6}[1 + 0.36\times 1.44] = \mathbf{25.3\ MN/m^2}$$

Example 2.4

A horizontal strut 2.5 m long is constructed from rectangular section steel, 50 mm wide by 100 mm deep, and mounted with pinned ends. The strut carries an axial load of 120 kN together with a uniformly distributed lateral load of 5 kN/m along its complete length. If $E = 200$ GN/m^2 determine the maximum stress set up in the strut.

Check the result using the approximate Perry method with

$$M_{\max} = M_0\left[\frac{P_e}{P_e - P}\right]$$

Solution

From eqn. (2.34)

$$M_{\max} = \frac{w}{n^2}\left(\sec\frac{nL}{2} - 1\right)$$

where $$n^2 = \frac{P}{EI} = \frac{120\times 10^3\times 12}{200\times 10^9\times 50\times 100^3\times 10^{-12}}$$

$$= 0.144$$

$$\therefore \qquad \frac{nL}{2} = \frac{2.5}{2}\sqrt{(0.144)} = 0.474 \text{ radian}$$

$$\therefore \qquad M_{\max} = \frac{5 \times 10^3}{0.144}(\sec 0.474 - 1)$$

$$= 34.7 \times 10^3(1.124 - 1) = 4.3 \times 10^3 \text{ Nm}$$

The maximum stress due to the axial load and the eccentricity caused by bending is then given by

$$\sigma_{\max} = \frac{P}{A} + \frac{My}{I}$$

$$= \frac{120 \times 10^3}{(0.1 \times 0.05)} + \frac{4.34 \times 10^3 \times 0.05 \times 12}{(50 \times 100^3)10^{-12}}$$

$$= 24 \times 10^6 + 51.6 \times 10^6$$

$$= \mathbf{75.6\ MN/m^2}$$

Using the approximate Perry method,

$$M_{\max} = M_0\left[\frac{P_e}{P_e - P}\right]$$

where $\qquad M_0 = \text{B.M. due to lateral load only} = \dfrac{wL^2}{8}$

But $\qquad P_e = \dfrac{\pi^2 EI}{L^2} = \dfrac{\pi^2 \times 200 \times 10^9}{2.5^2} \times \dfrac{(50 \times 100^3)10^{-12}}{12}$

$$= 1.316 \text{ MN}$$

$$\therefore \qquad M_{\max} = \frac{wL^2}{8}\left[\frac{P_e}{P_e - P}\right]$$

$$= \frac{5 \times 10^3 \times 2.5^2}{8}\left[\frac{1316 \times 10^3}{(1316 - 120)10^3}\right]$$

$$= 4.3 \times 10^3 \text{ Nm}$$

In this case, therefore, the approximate method yields the same answer for maximum B.M. as the full solution. The maximum stress will then also be equal to that obtained above, i.e. **75.6 MN/m²**.

Example 2.5

A hollow circular steel strut with its ends fixed in position has a length of 2 m, an outside diameter of 100 mm and an inside diameter of 80 mm. Assuming that, before loading, there is an initial sinusoidal curvature of the strut with a maximum deflection of 5 mm, determine the maximum stress set up due to a compressive end load of 200 kN. $E = 208$ GN/m^2.

Solution

The assumed sinusoidal initial curvature may be expressed alternatively in the complementary cosine form

$$y_0 = \delta_0 \cos\frac{\pi x}{L} \quad \text{(Fig. 2.17)}$$

Now when P is applied, y_0 increases to y and the central deflection increases from $\delta_0 = 5$ mm to δ.

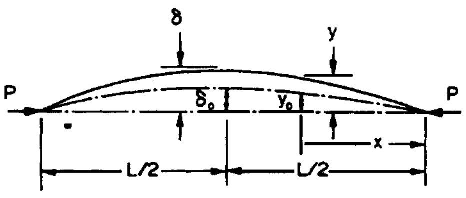

Fig. 2.17.

For the above initial curvature it can be shown that

$$\delta = \left[\frac{P_e}{P_e - P}\right]\delta_0$$

$$\therefore \qquad \text{maximum B.M.} = P\delta_0\left[\frac{P_e}{P_e - P}\right]$$

where P_e for ends fixed in direction only $= \dfrac{\pi^2 EI}{L^2}$

$$I = \frac{\pi}{64}(0.1^4 - 0.08^4) = \frac{\pi}{64}(1 - 0.41)10^{-4} = 2.896 \times 10^{-6}\text{m}^4$$

$$\therefore \qquad P_e = \frac{\pi^2 \times 208 \times 10^9 \times 2.89 \times 10^{-6}}{4} = 1.486 \text{ MN}$$

$$\therefore \qquad \text{maximum B.M.} = 200 \times 10^3 \times 5 \times 10^{-3}\left[\frac{1486 \times 10^3}{(1486 - 200)10^3}\right] = 1.16 \text{ kN m}$$

$$\therefore \qquad \text{maximum stress} = \frac{P}{A} + \frac{My}{I} = \frac{200 \times 10^3 \times 4}{\pi(0.1^2 - 0.08^2)} + \frac{1.16 \times 10^3 \times 0.05}{2.89 \times 10^{-6}}$$

$$= 70.74 \times 10^6 + 20.07 \times 10^6$$

$$= \mathbf{90.8\ MN/m^2}$$

Problems

2.1 (A/B). Compare the crippling loads given by the Euler and Rankine–Gordon formulae for a pin-jointed cylindrical strut 1.75 m long and of 50 mm diameter. (For Rankine–Gordon use $\sigma_y = 315$ MN/m^2; $a = 1/7500$; $E = 200$ GN/m^2.) [197.7, 171 kN.]

2.2 (A/B). In an experiment an alloy rod 1 m long and of 6 mm diameter, when tested as a simply supported beam over a length of 750 mm, was found to have a maximum deflection of 5.8 mm under the action of a central load of 5 N.

(a) Find the Euler buckling load when this rod is tested as a strut, pin-jointed and guided at both ends.
(b) What will be the central deflection of this strut when the material reaches a yield stress of 240 MN/m^2?

(Clue: maximum stress $= \dfrac{P}{A} \pm \dfrac{My}{I}$ where $M = P \times \delta_{max}$.) [74.8 N; 67 mm.]

2.3 (B) A steel strut is built up of two T-sections riveted back to back to form a cruciform section of overall dimensions 150 mm × 220 mm. The dimensions of each T-section are 150 mm × 15 mm × 110 mm high. The ends of the strut are rigidly secured and its effective length is 7 m. Find the maximum safe load that this strut can carry with a factor of safety of 5, given $\sigma_y = 315$ MN/m^2 and $a = 1/30000$ in the Rankine–Gordon formula. [192 kN.]

2.4 (B). State the assumptions made when deriving the Euler formula for a strut with pin-jointed ends. Derive the Euler crippling load for such a strut–the general equation of bending and also the solution of the differential equation may be assumed.

A straight steel rod 350 mm long and of 6 mm diameter is loaded axially in compression until it buckles. Assuming that the ends are pin-jointed, find the critical load using the Euler formula. Also calculate the maximum central deflection when the material reaches a yield stress of 300 MN/m^2 compression. Take $E = 200$ GN/m^2. [1.03 kN; 5.46 mm.]

2.5 (B). A steel stanchion 5 m long is to be built of two I-section rolled steel joists 200 mm deep and 150 mm wide flanges with a 350 mm wide × 20 mm thick plate riveted to the flanges as shown in Fig. 2.18. Find the spacing of the joists so that for an axially applied load the resistance to buckling may be the same about the axes XX and YY. Find the maximum allowable load for this condition with ends pin-jointed and guided, assuming $a = 1/7500$ and $\sigma_y = 315$ MN/m^2 in the Rankine formula.

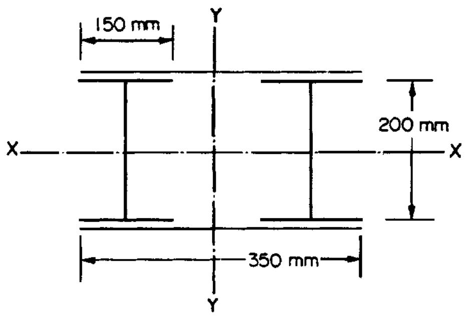

Fig. 2.18.

If the maximum working stress in compression σ for this strut is given by $\sigma = 135[1 - 0.005\, L/k]$ MN/m^2, what factor of safety must be used with the Rankine formula to give the same result? For each R.S.J. $A = 6250$ mm^2, $k_x = 85$ mm, $k_y = 35$ mm. [180.6 mm, 6.23 MN; 2.32.]

2.6 (B). A stanchion is made from two 200 mm × 75 mm channels placed back to back, as shown in Fig. 2.19, with suitable diagonal bracing across the flanges. For each channel $I_{xx} = 20 \times 10^{-6} m^4$, $I_{yy} = 1.5 \times 10^{-6}$ m^4, the cross-sectional area is 3.5×10^{-3} m^2 and the centroid is 21 mm from the back of the web.

At what value of p will the radius of gyration of the whole cross-section be the same about the X and Y axes? The strut is 6 m long and is pin-ended. Find the Euler load for the strut and discuss briefly the factors which cause the actual failure load of such a strut to be less than the Euler load. $E = 210$ GN/m^2. [163.6 mm; 2.3 MN.]

2.7 (B). In tests it was found that a tube 2 m long, 50 mm outside diameter and 2 mm thick when used as a pin-jointed strut failed at a load of 43 kN. In a compression test on a short length of this tube failure occurred at a load of 115 kN.

(a) Determine whether the value of the critical load obtained agrees with that given by the Euler theory.
(b) Find from the test results the value of the constant a in the Rankine–Gordon formula. Assume $E = 200$ GN/m^2. [Yes; 1/7080.]

2.8 (B). Plot, on the same axes, graphs of the crippling stresses for pin-ended struts as given by the Euler and Rankine–Gordon formulae, showing the variation of stress with slenderness ratio

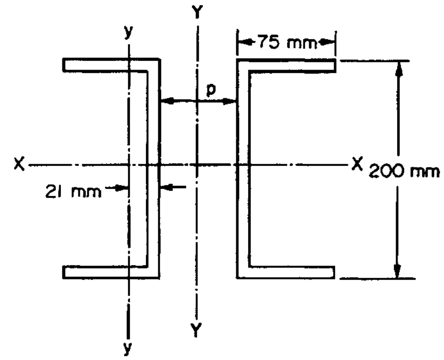

Fig. 2.19.

For the Euler formula use L/k values from 80 to 150, and for the Rankine formula L/k from 0 to 150, with $\sigma_y = 315$ MN/m^2 and $a = 1/7500$.

From the graphs determine the values of the stresses given by the two formulae when $L/k = 130$ and the slenderness ratio required by both formulae for a crippling stress of 135 MN/m^2. $E = 210$ GN/m^2.

[122.6 MN/m^2, 96.82 MN/m^2; 124,100.]

2.9 (B/C). A timber strut is 75 mm × 75 mm square-section and is 3 m high. The base is rigidly built-in and the top is unrestrained. A bracket at the top of the strut carries a vertical load of 1 kN which is offset 150 mm from the centre-line of the strut in the direction of one of the principal axes of the cross-section. Find the maximum stress in the strut at its base cross-section if $E = 9$ GN/m^2. [I.Mech.E.] [2.3 MN/m^2.]

2.10 (B/C). A slender column is built-in at one end and an eccentric load is applied at the free end. Working from first principles find the expression for the maximum length of column such that the deflection of the free end does not exceed the eccentricity of loading. [I.Mech.E.] [$\sec^{-1} 2/\sqrt{(P/EI)}$.]

2.11 (B/C). A slender column is built-in one end and an eccentric load of 600 kN is applied at the other (free) end. The column is made from a steel tube of 150 mm o.d. and 125 mm i.d. and it is 3 m long. Deduce the equation for the deflection of the free end of the beam and calculate the maximum permissible eccentricity of load if the maximum stress is not to exceed 225 MN/m^2. $E = 200$ GN/m^2. [I.Mech.E.] [4 mm.]

2.12 (B). A compound column is built up of two 300 mm × 125 mm R.S.J.s arranged as shown in Fig. 2.20. The joists are braced together; the effects of this bracing on the stiffness may, however, be neglected. Determine the safe height of the column if it is to carry an axial load of 1 MN. Properties of joists: $A = 6 \times 10^{-3}$ m^2; $k_{yy} = 27$ mm; $k_{xx} = 125$ mm.

The allowable stresses given by BS449: 1964 may be found from the graph of Fig. 2.9. [8.65 m.]

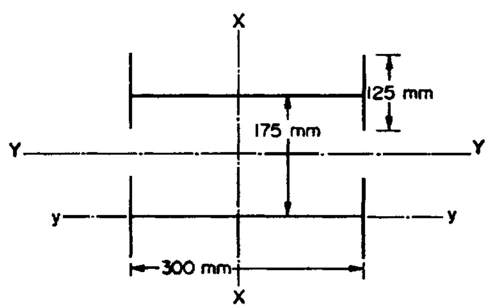

Fig. 2.20.

2.13 (B). A 10 mm long column is constructed from two 375 mm × 100 mm channels placed back to back with a distance h between their centroids and connected together by means of narrow batten plates, the effects of which may be ignored. Determine the value of h at which the section develops its maximum resistance to buckling.

Estimate the safe axial load on the column using the Perry–Robertson formula (a) with a load factor of 2, (b) with a factor of safety of 2. For each channel $I_{xx} = 175 \times 10^{-6}$ m^4, $I_{yy} = 7 \times 10^{-6}$ m^4, $A = 6.25 \times 10^{-3}$ m^2, $E = 210$ GN/m^2 and yield stress = 300 MN/m^2. Assume $\eta = 0.003\ L/k$ and that the ends of the column are effectively pinned. [328 mm; 1.46, 1.55 MN.]

2.14 (B). (a) Compare the buckling loads that would be obtained from the Rankine–Gordon formula for two identical steel columns, one having both ends fixed, the other having pin-jointed ends, if the slenderness ratio is 100.

(b) A steel column, 6 m high, of square section 120 mm × 120 mm, is designed using the Rankine–Gordon expression to be used as a strut with both ends pin-jointed.

The values of the constants used were $a = 1/7500$, and $\sigma_c = 300$ MN/m^2. If, in service, the load is applied axially but parallel to and a distance x from the vertical centroidal axis, calculate the maximum permissible value of x. Take $E = 200$ GN/m^2. [7.4; 0.756 m.]

2.15 (B). Determine the maximum compressive stress set up in a 200 mm × 60 mm I-section girder carrying a load of 100 kN with an eccentricity of 6 mm. Assume that the ends of the strut are pin-jointed and that the overall length is 4 m.

Take $I = 3 \times 10^{-6}$ m^4; $A = 6 \times 10^{-3}$ m^2 and $E = 207$ GN/m^2. [25.4 MN/m^2.]

2.16 (B). A slender strut, initially straight, is pinned at each end. It is to be subjected to an eccentric compressive load whose line of action is parallel to the original centre-line of the strut.

(a) Prove that the central deflection y of the strut, relative to its initial centre-line, is given by the expression

$$y = e\left[\sec \frac{1}{2}\sqrt{\left(\frac{PL^2}{EI}\right)} - 1\right]$$

where P is the applied load, L is the effective length of the strut, e is the eccentricity of the line of action of the load from the initially straight strut axis and EI is the flexural rigidity of the strut cross-section.

(b) Using the above formula, and assuming that the strut is made of a ductile material, show that, for a maximum compressive stress, σ, the value of P is given by the expression

$$P = \frac{\sigma A}{\dfrac{he}{k^2}\sec \dfrac{1}{2}\sqrt{\left(\dfrac{PL^2}{EI}\right)} + 1}$$

the symbols A, h and k having their usual meanings.

(c) Such a strut, of constant tubular cross-section throughout, has an outside diameter of 64 mm, a principal second moment of area of 52×10^{-8}m^4 and a cross-sectional area of 12.56×10^{-4}m^2. The effective length of the strut is 2.5 m. If $P = 120$ kN and $\sigma = 300$ MN/m^2, determine the permissible value of e. Take $E = 200$ GN/m^2. [B.P.] [6.25 mm.]

2.17 (C). A strut of length L has each end fixed in an elastic material which can exert a restraining moment μ per radian. Prove that the critical load P is given by the equation

$$P + \mu\sqrt{\left(\frac{P}{EI}\right)}\tan \frac{L}{2}\sqrt{\left(\frac{P}{EI}\right)} = 0$$

The designed buckling load of a 1 m long strut, assuming the ends to be rigidly fixed, was 2.5 kN. If, during service, the ends were found to rotate with each mounting exerting a restraining moment of 1 kN m per radian, show that the buckling load decreases by 20%. [C.E.I.]

2.18 (C). A uniform elastic bar of circular cross-section and of length L, free at one end and rigidly built-in at the other end, is subjected to a single concentrated load P at the free end. In general the line of action of P may be at an angle θ to the axis of the bar ($0 < \theta < \pi/2$) so that the bar is simultaneously compressed and bent. For this general case:

(a) Show that the deflection at the free end is given by

$$\delta = \tan\theta\left\{\left(\frac{\tan mL}{m} - L\right)\right\}$$

(b) Hence show that as $\theta \to \pi/2$, then $\delta \to PL^3/3EI$
(c) Show that when $\theta = 0$ no deflection unless P has certain particular values.

Note that in the above, m^2 denotes $P\cos\theta/EI$.
The following expression may be used in part (b) where appropriate:

$$\tan\alpha = \alpha + \frac{\alpha^3}{3} + \frac{2\alpha^5}{15}$$ [City U.]

2.19 (C). A slender strut of length L is encastré at one end and pin-jointed at the other. It carries an axial load P and a couple M at the pinned end. If its flexural rigidity is EI and $P/EI = n$, show that the magnitude of the couple at the fixed end is

$$M\left[\frac{nL - \sin nL}{nL\cos nL - \sin nL}\right]$$

What is the value of this couple when (a) P is one-quarter the Euler critical load and (b) P is zero?
[U.L.] [0.571 M, 0.5 M.]

2.20 (C). An initially straight strut of length L has lateral loading w per metre and a longitudinal load P applied with an eccentricity e at both ends.

If the strut has area A, second moment of area I, section modulus Z and the end moments and lateral loading have opposing effects, find an expression for the central bending moment and show that the maximum stress at the centre will be equal to

$$\frac{P}{A} + \frac{\left(Pe - \dfrac{wEI}{P}\right)\sec\dfrac{L}{2}\sqrt{\left(\dfrac{P}{EI}\right)} + \dfrac{wEI}{P}}{Z}$$ [U.L.]