Chapter 6

Principles of Work and Energy

6.1 Introduction

Theoretically, the problems of continuous solids and structures can be formulated, solved or approximated, by the mechanics of the preceding chapters. These provide the differential equations of equilibrium (or motion) governing stresses, the kinematical equations relating displacements and strains, and then certain constitutive equations relating the stresses and strains. That is the so-called “vectorial mechanics” (basic variables are vectors), also termed Newtonian. An alternative approach is based upon variations of work and energy. The latter is the so-called “analytical mechanics,” or Lagrangean; both terms may be traced to J. L. Lagrange and his treatise, “Mécanique Analytique” [102]. In all instances, the alternative approach must provide the same description of the mechanical system. However, in many instances the energetic approach has advantages and frequently provides the most powerful means for effective approximations. This last attribute is especially important in the modern era of digital computation which facilitates the approximation of continuous bodies by discrete systems (e.g., finite elements).

Our treatment of the principles and methods is necessarily limited. We confine our attention to the concepts and procedures as applied to equilibrated systems. These are given in generality; principles, associated theorems, and necessary functionals apply to finite deformations of continuous bodies. In most instances, these can be readily extended to dynamic systems.

Although our primary concern is the analysis of continuous bodies, the salient features of the principles are most evident in simpler discrete systems. Then too, we recognize the important applications to discrete models of continuous bodies. Accordingly, the principles are initially formulated here for discrete mechanical systems and later reformulated for the continuous body. Specifically, the work or energy functions of the discrete
variables are replaced by the *functionals* of the *continuous* functions; mathematical forms, but not concepts, differ. Those principal concepts follow our presentation of the essential terminology.

### 6.2 Historical Remarks

The basic concept of virtual work has been traced to Leonardo da Vinci (1452–1519). That notion was embodied in the numerous subsequent works of G. W. von Leibnitz (1646–1716), J. Bernoulli (1667–1748), and L. Euler (1707–1783); these and others embraced the concept of varying energy, both potential and kinetic. J. L. Lagrange (1736–1813) extended the concept of virtual work to dynamical systems via the D’Alembert’s principle (1717–1783). Further generalization was formulated by W. R. Hamilton (1805–1865). An excellent historical account was given by G. Æ. Oravas and L. McLean [103], [104]. A most interesting philosophical/mathematical treatment and a historical survey are contained in the text of C. Lanczos [29]. Practical applications, as well as theoretical and historical background, are given in the book by H. L. Langhaar [10]. A rudimentary treatment may be found in the previous texts by G. A. Wempner [105], [106].

### 6.3 Terminology

**Generalized Coordinates**

The *configuration* of a mechanical system is determined by the positions of all the particles that comprise the system. The configuration of a discrete system consists of a finite number of particles or rigid bodies, and consequently, the configuration is defined by a finite number of real variables called *generalized coordinates*. An example is the angle which defines the configuration of a pendulum. A continuous body is conceived as an infinite collection of particles and, consequently, its configuration must be specified by an infinity of values, a continuous vector function.

Any change in the configuration of a system is referred to as a *displacement* of the system. To remove any ambiguity when the term “displacement” is used in such context, we define the magnitude of the displacement as the largest distance traveled by any of the particles comprising the system. In addition, we suppose that any motion occurs in a continuous fashion; that is, the distance traveled vanishes with the time of travel.
The generalized coordinates are assumed regular in the sense that any increment in the coordinate is accompanied by a displacement of the same order of magnitude and vice versa.

**Degrees of Freedom**

The number of degrees of freedom is the minimum number of generalized coordinates. For example, a particle lying on a table has two degrees of freedom; a ball resting on a table has five because three angles are also needed to establish its orientation.

**Constraints**

Any geometrical condition imposed upon the displacement of a system is called a constraint. A ball rolling upon a table has a particular kind of constraint imposed by the condition that no slip occurs at the point of contact. The latter is a nonintegrable differential relation between the five coordinates; such conditions are called nonholonomic constraints.

**Virtual Displacements**

At times, it is convenient to imagine small displacements of a mechanical system and to examine the work required to effect such movement. To be meaningful, such displacements are supposed to be consistent with all constraints imposed upon the system. Since these displacements are hypothetical, they are called virtual displacements.

**Work**

If \( f' \) denotes the resultant force upon a particle and \( V \) the displacement vector, then the work performed by \( f' \) during an infinitesimal displacement \( \delta V \) is

\[
\delta w = f' \cdot \delta V.
\]  

(6.1a)

If \( v \equiv dV/dt \), then the work done in time \( \delta t \) is

\[
\delta w = f' \cdot v \, \delta t.
\]  

(6.1b)

In the time interval \( (t_0, t_1) \), the work done by \( f' \) is

\[
\Delta w \equiv \int_{t_0}^{t_1} f' \cdot v \, dt.
\]  

(6.2)

**Generalized Forces**

In general, the particles comprising a system do not have complete freedom of movement but are constrained to move in certain ways. For exam-
ple, a simple pendulum consists of a particle at the end of a massless link which constrains the particle to move on a circular path; the position of the particle is defined by one generalized coordinate, the angle of rotation of the link. In a discrete mechanical system, the displacements of all particles can be expressed in terms of generalized coordinates. If \( q_i \) \((i = 1, \ldots, n)\) denotes a regular generalized coordinate for a system with \( n \) degrees of freedom, then the position of a particle \( N \) is given by a vector \( \mathbf{R}_N(q_1, \ldots, q_n) \) and the incremental displacement of the particle is

\[
\delta \mathbf{V}_N = \sum_{i=1}^{n} a^i_N \delta q_i, \quad a^i_N \equiv \frac{\partial \mathbf{R}_N}{\partial q_i} = \mathbf{R}_{N,i}.
\]  

(6.3a, b)

If \( \mathbf{f}^N \) denotes the force acting upon the particle \( N \), then the work done during the displacement is

\[
\delta w_N = \mathbf{f}^N \cdot \delta \mathbf{V}_N \quad \text{(no summation on } N).\]

The work performed by all forces acting upon all particles of the system is

\[
\delta W = \sum_N \mathbf{f}^N \cdot \delta \mathbf{V}_N
\]

\[
= \sum_{i=1}^{n} \left( \sum_N \mathbf{f}^N \cdot a^i_N \right) \delta q_i.
\]

If we define

\[
Q^i \equiv \sum_N \mathbf{f}^N \cdot a^i_N,
\]

then

\[
\delta W = \sum_{i=1}^{n} Q^i \delta q_i. \quad (6.4)
\]

The quantity \( Q^i \) is termed a \textit{generalized force}. The modifier “generalized” is inserted because the quantity need not be a physical force; for example, if the angle of rotation is the generalized coordinate for a pendulum, then the moment about the pivot is the generalized force.
6.4 Work, Kinetic Energy, and Fourier’s Inequality

Law of Kinetic Energy

According to Newton’s law, the force on a particle of mass \( m \) moving with velocity \( v \) is

\[
f' = m \frac{dv}{dt}.
\]

(6.5)

It follows from (6.1b) and (6.5) that

\[
\delta w = m \frac{dv}{dt} \cdot v \delta t = \frac{d}{dt} \left( \frac{1}{2} mv^2 \right) \delta t.
\]

The parenthetical term is the kinetic energy of the particle

\[
\tau = \frac{1}{2} mv^2.
\]

(6.6)

Therefore,

\[
\delta w = \delta \tau.
\]

(6.7)

Equation (6.7) states that the increment of work is equal to the increment of kinetic energy.

Viewing a mechanical system as a collection of particles and summing the work done upon all particles, we conclude that the work of all forces upon a mechanical system equals the increase of the kinetic energy of the system:

\[
\delta W \equiv \sum \delta w = \sum \delta \tau \equiv \delta T,
\]

(6.8)

where the summation extends over all particles of the system and \( T \) denotes the kinetic energy of the entire system.

If (6.8) is integrable, i.e., \( \delta T = dT \), then over a time interval we obtain

\[
\Delta W = \int_{t_0}^{t_1} dT = \Delta T.
\]

(6.9)

Equation (6.9) is called the law of kinetic energy. It is important to realize that the work \( W \) includes the work of internal forces as well as the work of forces exerted by external agencies.

Fourier’s Inequality

According to the law of kinetic energy (6.9), the kinetic energy of a mechanical system cannot increase unless positive work is performed. If
the system is at rest and there is no way that it can move such that the net work of all forces is positive, then the system must remain at rest. Stated otherwise, if the work performed during every virtual displacement is negative or zero, the system must remain at rest; that is, a sufficient condition for equilibrium is

\[ \Delta W \leq 0 \]  \hspace{1cm} (6.10)

for every small virtual displacement that is consistent with the constraints. The inequality (6.10) is known as *Fourier’s inequality*.

### 6.5 The Principle of Virtual Work

According to Newton’s first law, if the particle is at rest (or in steady motion \( \dot{v} = 0 \)) then the resultant force vanishes (\( \mathbf{f} = 0 \)). During any infinitesimal movement of the particle, no work is done; that is,

\[ \delta W = \mathbf{f} \cdot \delta \mathbf{V} = 0. \]

Likewise, if we suppose that all particles of an equilibrated mechanical system are given infinitesimal displacements, then the work performed by all forces upon the system vanishes:

\[ \delta W = 0. \]  \hspace{1cm} (6.11)

Since the equality of (6.10) is a sufficient condition for equilibrium, equation (6.11) is both necessary and sufficient provided that the virtual displacements are consistent with any constraints imposed upon the system; the condition expresses the *principle of virtual work*. According to (6.11), the increment of virtual work vanishes if the system is in equilibrium and the movement is consistent with constraints. If the generalized coordinates \( q_i \) of (6.4) are independent, then the right side of (6.4) vanishes for arbitrary \( \delta q_i \); it follows that

\[ Q^i = 0. \]  \hspace{1cm} (6.12)

The foregoing statement of the principle cannot be taken without qualification. Specifically, both forces and displacements are supposed to be continuous variables; abrupt discontinuities are inadmissible. The systems of Figure 6.1 illustrate the point. The roller of Figure 6.1a rests in a V-groove; any movement, as that from \( O \) to \( O' \), requires work. Note that the force of contact at \( A \) vanishes *abruptly*; the path of admissible displacement exhibits an *abrupt* corner. Moreover, work and displacement are of the same
order of magnitude. The system of Figure 6.1b conforms to our criteria. A small motion along the smooth path requires work of higher order. Stated otherwise, the work $\Delta W$ and displacement $\Delta s$ must not be of the same order of magnitude.⁷ To be more specific, let $y$ denote the vertical position of the ball and $s$ denote distance on the path $OO'$ (see Figure 6.1). The work of first order performed by the weight $f$ of the roller in Figures 6.1a, b follows:

**Case a:** $\Delta W = -f \frac{dy}{ds} \Delta s, \quad \frac{\Delta W}{\Delta s} = -f \frac{dy}{ds} \neq 0$,

**Case b:** $\Delta W = -f \frac{dy}{ds} \Delta s, \quad \frac{\Delta W}{\Delta s} = 0$.

The latter (b) follows since $dy/ds = 0$ at the equilibrium position, i.e., at the bottom of the trough. A small motion consistent with the constraint (rolls along the smooth path) requires work of higher order (no work of the first order in the displacement). The former (a) violates the requirement, since $dy/ds \neq 0$. It follows that the system of Figure 6.1b conforms to the criteria, the system of Figure 6.1a does not.

**Internal and External Forces**

In the analyses of structures, it is particularly convenient to classify the forces as internal or external forces. An internal force is an interaction between parts of the system while an external force is exerted by an external agency.

We signify the work of internal and external forces with minuscule (lowercase) prefixes “in” and “ex,” respectively. The work of all forces is the sum
\[ W = W_{in} + W_{ex}. \] (6.13)
According to the principle of virtual work, equation (6.11),
\[ \delta (W_{in}) = -\delta (W_{ex}). \] (6.14)

6.6 Conservative Forces and Potential Energy

A force is said to be conservative if the work it performs upon a particle during transit from an initial position \( P_0 \) to another position \( P \) is independent of the path traveled. If the force is conservative, then it does no work during any motion which carries the particle along a closed path terminating at the point of origin \( P_0 \). The work performed depends upon the initial and current positions \( P_0 \) and \( P \). Since we are concerned only with changes of the current position, the initial position is irrelevant and, therefore, we regard the work as a function of the current position, that is,
\[ w = \int_{P_0}^{P} \mathbf{f} \cdot d\mathbf{V} = w(P). \]

The potential of the force \( \mathbf{f} \) is denoted \( v(P) \);
\[ v(P) \equiv -w(P). \] (6.15)

In view of the negative sign, it may be helpful to regard \( v \) as the work that you would do upon the particle if you held the particle (in equilibrium by exerting force \( -f \)) and transported it (slowly) to the position \( P \). If \( P \) is defined by Cartesian coordinates \( X_i \), then the differential of (6.15) is
\[ dv = \frac{\partial v}{\partial X_i} dX_i \equiv -f^i dX_i. \]

It follows that the conservative force derives from the potential
\[ f^i = -\frac{\partial v}{\partial X_i}. \] (6.16)
Two common examples of conservative forces are a gravitational force and a force exerted by an elastic spring. If one moves an object from the floor to a table, the work expended is independent of the path taken; it depends only on the change of elevation. If one grasps one end of the spring and changes its position, while the other end remains fixed, then the work performed is independent of the path; it depends only on the change of length.

If the configuration of a conservative mechanical system is defined by generalized coordinates $q_i$ ($i = 1, \ldots, n$), and if $W$ is the work of the (conservative) forces acting upon the entire system, then there exists a potential energy $V$ such that

$$-W = V(q_i).$$

The differential of work done by the conservative forces is

$$dW = -dV,$$  \hspace{1cm} (6.17a)

$$= - \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} dq_i.$$  \hspace{1cm} (6.17b)

In accordance with (6.4), the generalized forces are

$$Q^i = - \frac{\partial V}{\partial q_i}.$$  \hspace{1cm} (6.18)

The potential of internal and external forces are denoted by $U$ and $\Pi$, that is,

$$W_{in} = -U, \quad W_{ex} = -\Pi.$$  \hspace{1cm} (6.19a, b)

The generalized internal and external forces are

$$F^i = - \frac{\partial U}{\partial q_i}, \quad P^i = - \frac{\partial \Pi}{\partial q_i}.$$  \hspace{1cm} (6.20a, b)

The total potential energy of the system is the sum

$$V = U + \Pi.$$  \hspace{1cm} (6.21)
6.7 Principle of Stationary Potential Energy

A mechanical system is conservative if all forces, internal and external, are conservative. In this case, no mechanical energy is dissipated, that is, converted to another form such as heat. Then the principle of virtual work asserts that

\[ dV = \sum_{i=1}^{n} \frac{\partial V}{\partial q_i} dq_i = 0. \]  

(6.22)

If the coordinates are independent, then

\[ Q_i = -\frac{\partial V}{\partial q_i} = 0. \]  

(6.23)

Since the equations (6.23) are the conditions which render the function \( V \) stationary, the principle of virtual work becomes the principle of stationary potential energy. In view of (6.21), (6.23) has the alternative form:

\[ \frac{\partial U}{\partial q_i} = -\frac{\partial \Pi}{\partial q_i}. \]  

(6.24)

In accordance with (6.20b) and (6.24), the external force \( P^i \) upon the equilibrated system satisfies the equation:

\[ P^i = \frac{\partial U}{\partial q_i}. \]  

(6.25)

6.8 Complementary Energy

Equation (6.25) expresses the force \( P^i \) in terms of the coordinates \( q_i \) and, if the second derivatives exist, then

\[ dP^i = \sum_{j=1}^{n} \frac{\partial^2 U}{\partial q_i \partial q_j} dq_j. \]

If the determinant of the second derivatives does not vanish, then these equations can be inverted, such that

\[ q_i = q_i(P^i). \]  

(6.26)
In view of (6.26), we can define a complementary energy,

\[ C(P^i) \equiv \sum_{i=1}^{n} P^i q_i - U. \]  

(6.27)

Then

\[ \frac{\partial C}{\partial P^j} = q_j + \sum_{i=1}^{n} \left( P^i - \frac{\partial U}{\partial q_i} \right) \frac{\partial q_i}{\partial P^j}. \]

Equation (6.25) holds for the equilibrated system; then in view of (6.25),

\[ q_i = \frac{\partial C}{\partial P^i}. \]  

(6.28)

Note that the variables \( q_i \) are related to the complementary energy \( C(P^i) \) as the variables \( P^i \) are related to the potential \( U(q_i) \). The transformation which reverses the roles of the complementary energies (\( U \) and \( C \)) and the variables (\( P^i \) and \( q_i \)) is known as a Legendre transformation.

### 6.9 Principle of Minimum Potential Energy

The principle of virtual work asserts the vanishing of the virtual work in an infinitesimal virtual displacement, that is, \( \delta W = 0 \). Fourier’s inequality asserts that a mechanical system does not move from rest (to an adjacent or distant configuration) unless it can move in some way that the active forces do positive work; if \( \Delta W \leq 0 \) the system must remain at rest.

Let us now distinguish between the inequality \( \Delta W < 0 \) and the equality \( \Delta W = 0 \), if the displacements are other then infinitesimals of the first order. The inequality implies that the system can be moved only if an outside agency does positive work upon the system. The equality means that the system can be slowly transported without doing work (“slowly,” because no energy is supplied to increase the kinetic energy). The simple system of Figure 6.2 consisting of a ball resting (a) in a cup and (b) on a horizontal surface illustrates the difference. The active force on the ball is the gravitational force, the weight. The distances of the center from the reference point \( P \) is a suitable generalized coordinate if each ball is to roll upon its respective surface. Both are in equilibrium in position \( P(\delta W = 0) \), but \( \Delta W < 0 \) in case (a) (from \( P \) to \( S \)) and \( \Delta W = 0 \) in case (b). In case (b), the position \( P \) is one of neutral equilibrium. In general, if \( \Delta W < 0 \) for sufficiently small displacements, then the system is stable. The
qualification of “sufficiently small” is needed to discount the circumstances in which negative work is performed initially but subsequently exceeded by a greater amount of positive work. For example, if the ball of Figure 6.2a is transported from $P$ to $Q$ the net work done by the gravitational force is positive, but some negative work is done as it is transported to the crest of the hill.

If the system is conservative, then the work done by all forces upon the system is equivalent to the decrease in the total potential energy, that is, $\Delta W = -\Delta V$. Then the requirement for stable equilibrium is $\Delta V > 0$.

In words, an equilibrium configuration of a conservative mechanical system is regarded as stable if the potential energy is a proper minimum. However, the engineer must exercise great care in his application of this criterion, because a physical system may be more or less stable as demonstrated by the ball of Figure 6.2a; if the hill between the valley $P$ and the lower valley $Q$ is very low, then a small disturbance may render the weak stability at $P$ insufficient to prevent the excursion to $Q$.

The principle of minimum potential is a cornerstone in the theory of elastic members and structures. Specifically, most criteria and analyses for the stability of equilibrium are founded upon this principle. The criterion of E. Trefftz [25] establishes the critical state for most structural systems. The criteria and analyses of W. T. Koiter [107] provide additional insights to the behavior at the critical state. Additionally, Koiter’s work offers means to predict the abrupt, often catastrophic, “snap-through buckling” of certain shells and the related effects of imperfections. Here, we present the essential arguments and exhibit those criteria in the context of a discrete system. These are readily extended to continuous bodies; the algebraic functions of

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the discrete variables (e.g., generalized coordinates $q_i$) are supplanted by
the functionals of the continuous functions [e.g., displacements $V_i(\theta)$].

Sources for further study are found in the work of J. W. Hutchinson and
W. T. Koiter [108]. A clear presentation, practical applications and references,
are contained in the book by H. L. Langhaar [10]. The monograph by
Z. P. Bázant and L. Cedolin [109] offers a comprehensive treatment of the
stability theory of structures. The book covers subjects relevant to many
branches of engineering and the behavior of materials. It presents alterna-
tive methods of analysis, investigates the stability of various structural
elements, and contains practical applications; the second part is devoted
to the stability of inelastic systems. The monograph by H. Leipholz [110]
gives an introduction to the stability of elastic systems, emphasizes the dy-
namic aspects of instability, and presents methods of solution; other topics
include nonconservative loadings and stochastic aspects.

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6.10 Structural Stability

The principle of minimum potential (Section 6.9) provides a basis for the
analyses of structural stability. Some applications reveal certain practical
implications and responses. To that end, we examine the simple linkage
depicted in Figure 6.3. That system has but one degree of freedom; the
lateral displacement $w$ of the joint $B$. Although simple, the system exhibits
important attributes of complex structures.

The two rigid bars, $AB$ and $BC$, are joined by the frictionless pins at
$A$, $B$, and $C$. Pin $C$ inhibits translation; $A$ constrains movement to the
straight line $AC$. Extensional springs at $B$ resist lateral movement by the
force $F = kw$ and the torsional spring at $B$ resists relative rotation ($2\theta$)
by a couple $C = \beta(2\theta)$. This simple system exhibits a behavior much as an
elastic column (spring $\beta$ simulates bending resistance) with lateral restraint
(spring $k$ simulates the support of a lateral beam). We now explore the
response to the thrust force $P = \text{constant} \ (\text{as a weight})$.

The potential energy of the displaced system of Figure 6.3b consists of
the potential $\Pi$ of the “dead” load $P^t$ and the internal energy $U$ of the
springs ($k$ and $\beta$):

$$\mathcal{V} = \Pi + U = 2Pl \cos \theta + \frac{kl^2}{2} \sin^2 \theta + \frac{\beta}{2} (2\theta)^2. \quad (6.29)$$

^1Note that any constant can be added to the potential energy without affecting our
analysis, since we are only concerned with variations in potential energy; we could as
well set $\Pi = -2Pl(1 - \cos \theta)$. 

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By the principle of stationary potential, the system is in equilibrium if

\[ \delta V = (-2Pl \sin \theta + k l^2 \sin \theta \cos \theta + 4\beta \theta) \delta \theta = 0. \]  

(6.30)

The condition must hold for arbitrary \( \delta \theta \); therefore, the equation of equilibrium follows:

\[ -2Pl \sin \theta + k l^2 \sin \theta \cos \theta + 4\beta \theta = 0. \]

Evidently, \( \theta = 0 \) is an equilibrium state for all choices of \( P, l, k, \beta \). Additionally, the equation is satisfied for nonzero \( \theta \), if

\[ P = \frac{kl}{2} \cos \theta + \frac{2\beta}{l} \frac{\theta}{\sin \theta}. \]  

(6.31)

Note that \( P(\theta) \) in (6.31) is an even function of the variable \( \theta \); it intersects the ordinate at

\[ P \equiv P_{cr} = \frac{kl}{2} + \frac{2\beta}{l}. \]  

(6.32)
Figure 6.4 Stable versus unstable states of energy

Figure 6.4 displays plots of the dimensionless load \( \frac{P}{P_{cr}} \) for various values of the parameters \( k \) and \( \beta \). The point \( P \) at the critical load \( \left[ \frac{P}{P_{cr}} = 1 \right] \) is called a bifurcation point: The branches of equilibrium states (6.31) intersect the stem \( OP \) at point \( P \).

First, let us consider the situation \( k = 0 \), the linkage without lateral support. The branch \( PS \) is initially normal to the stem, but henceforth has positive slope. The states on the stem \( OP \) are stable, unstable above \( P \). States on the branch \( PS \) are stable, since an ever increasing load is required to cause additional deflection. Practically, the “structure” fails at \( P \geq P_{cr} \), because slight increases in loading cause excessive deflection. A column, for example, is said to buckle at the critical load.

Now, consider the system when \( kl^2 = 8\beta \). The perfect system is, strictly speaking, stable along the stem \( OP \), unstable above \( P \). The state at the critical load \( P_{cr} \) or slightly below, say \( P = 0.8P_{cr} \), are especially interesting. As the unsupported linkage, the system does sustain a load \( P < P_{cr} \) in the straight configuration to the critical load \( P_{cr} \); at the critical load, the system snaps-through to the configuration of \( Q \). The responses of this system are best explained by examining the potentials at the specific loads. At the critical load, the potential traces the curve \( P'Q' \). Like the ball at the top of the hill (e.g., \( P' \)), the system moves to the valley (e.g., \( Q' \)), a state \( Q \) of
lower energy.

The behavior of the latter system \((kl^2 = 8\beta)\) under load \(P = 0.8P_{cr}\) is more interesting. The potential traces the curve \(A'B'C'\). We note that the state \(A\) is a stable one; however, the potential “valley” at \(A'\) is shallow. A slight disturbance can cause the system to snap-through to the state \(C\). The system does not rest at the equilibrium state \(B\), since it is unstable; \(B'\) is at the top of a hill. The system comes to rest at \(C\), where the potential has a valley at \(C'\). Some structures, especially thin shells, are subject to snap-through buckling. Note that slight imperfections or disturbances can cause the snap-through at loads far less than the theoretical critical load. The questions of stability, or instability, and the effects of imperfections are pursued in the subsequent sections.

The preceding example exhibits stability or instability at the critical load, depending on the relative magnitudes of the parameters. An examination of the potential indicates that this system is stable at the critical load, \(\Delta V|_{P_{cr}} > 0\), if, and only if,

\[
\frac{3kl^2}{4\beta} < 1.
\]

This result can also be ascertained by examining the geometrical properties of the branch \(P(\theta)\) at \(P_{cr}, \theta = 0\). At the critical load, the trace \(P(\theta)\) goes from concave to convex, from valley to hill.

### 6.11 Stability at the Critical Load

The simple example of Section 6.10 indicates that a mechanical system or structure can exhibit quite different responses (i.e., stable versus unstable) when subjected to critical loads. The energy criteria for such critical loads were established by E. Trefftz [25]. Criteria for stability and behavior at the critical load were developed by W. T. Koiter. Here, we present underlying concepts and some consequences of Koiter’s criteria; these are set in the context of a discrete system, but apply as well to a continuous body via the alternative mathematics, i.e., functionals replace functions, integrals replace summations, etc.

All loads upon the system are assumed to increase in proportion and, therefore, the magnitude is given by a positive parameter \(\lambda\). A configuration of the system is defined by \(N\) generalized coordinates \(q_i (i = 1, \ldots, N)\). As the loading parameter is increased from zero, the equilibrium states trace a path in a configuration-load space. For example, a system with two degrees of freedom \((q_1, q_2)\) follows a path in the space \((q_1, q_2; \lambda)\) of Figure 6.5a or 6.5b.

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The point $P$ of Figure 6.5a or 6.5b is a critical state, characterized by the existence of neighboring states which are not uniquely determined by an increment of the load. At the critical point $P$ of Figure 6.5a the path $OP$ forms two branches, $PR$ and $PQ$. The branch $PQ$ may ascend or descend, or the tangent $\hat{W}$ may be normal to the $\lambda$ axis. At the point $P$ of Figure 6.5b, if the smooth curve has a tangent $\hat{W}$ normal to the $\lambda$ axis, and if $S$ denotes arc length along the path, then the path $PQ$ at $P$ is characterized by the condition $d\lambda/dS = 0$; in words, the system tends to move from $P$ with no increase of load.

The critical state of Figure 6.5a occurs at a bifurcation point; two paths of equilibrium emanate from $P$. However, the path $PR$ represents unstable paths which cannot be realized. Actually, the system tends to move along $PQ$. If $PQ$ is an ascending path, then additional loading is needed, and the system is said to be stable at the critical state. In actuality, a very slight increment is usually enough to cause an unacceptable deflection and the system is said to buckle. If the curve $PQ$ is descending, the system collapses under the critical load $\lambda^*$. 

The path of Figure 6.5b is entirely smooth, but reaches a so-called limit point $P$. The state of $P$ is again critical in the sense that the tangent $\hat{W}$ is normal to the $\lambda$ axis. At $P$, the system tends to move under the critical load $\lambda^*$. Since the path descends from $P$ to $Q$, the system exhibits a “snap-through buckling”; it essentially collapses to a much disfigured state.
By our remarks, instability is characterized by the advent of excessive
deflections which are produced by a critical load \( \lambda^* \). However, the stabil-
ity of a conservative system can be characterized by an energy criterion:
the conservative mechanical system is in stable equilibrium if the potential
energy is a proper minimum, unstable if any adjacent state has a lower
potential. Let us apply the energy criterion at the critical state.

We presume that the potential energy can be expanded in a power series
about the critical state. If \((q_i; \lambda)\) defines a state of equilibrium, \( u_i \equiv \Delta q_i \)
defines a displacement from the reference state, and \( \Delta V \) the change of
potential caused by the displacement, then

\[
\Delta V = \bar{A}_i u_i + \frac{1}{2} A_{ij} u_i u_j + \frac{1}{3!} A_{ijk} u_i u_j u_k + \frac{1}{4!} A_{ijkl} u_i u_j u_k u_l + \cdots, \tag{6.33}
\]

where

\[
A_{ij...} = A_{ij...}(\lambda). \tag{6.34}
\]

With no loss of generality, the coefficients are treated as entirely symmetric
in their indices.

Since the state is a state of equilibrium, in accordance with the principle
of virtual work,

\[
\bar{A}_i = 0. \tag{6.35}
\]

If the quadratic term of (6.33) does not vanish identically, then it dom-
inates for small enough displacement. It follows that the state is stable if

\[
V_2(u_i) \equiv \frac{1}{2} A_{ij} u_i u_j > 0. \tag{6.36}
\]
The state is critical if

\[
\frac{1}{2} A_{ij} u_i u_j \geq 0. \tag{6.37}
\]
In words, the state is critical if there exists one (or more), nonzero displacement(s) \( u_i \) which causes the quadratic term to vanish, that is,

\[
V_2(\bar{u}_i) \equiv \frac{1}{2} A_{ij} \bar{u}_i \bar{u}_j = 0. \tag{6.38}
\]
The displacement \( \bar{u}_i \) is a buckling mode.

A minimum is characterized by a stationary condition. Here, the required
minimum of \( V_2(u_i) \) is determined by the stationary criterion of E. Tre-
fftz [25]. For an arbitrary variation \( \delta u_i \),

\[
\delta V_2 = A_{ij} u_i \delta u_j = 0. \tag{6.39}
\]
It follows that the buckling mode \( \bar{u}_i \) is a nontrivial solution of the equations:

\[
A_{ij} \bar{u}_i = 0. \tag{6.40}
\]
The homogeneous system has a nontrivial solution if, and only if, the determinant of coefficients vanishes

$$|A_{ij}(\lambda)| = 0.$$  \hfill (6.41)

The least solution of (6.41) determines the critical load $\lambda^*$. Let

$$\bar{u}_i = \epsilon W_i,$$  \hfill (6.42)

where $W_i$ are components of the unit vector in our $N$-dimensional space of $q_i$; that is,

$$W_i W_i = 1.$$  \hfill (6.43)

The parameter $\epsilon$ measures the magnitude of an excursion from the critical state and $\hat{W}_i$ defines the direction of the buckling.

Let us consider a movement along the path emanating from the critical point $P$. In the plane of $(q_1, q_2)$, we see a path as shown in Figure 6.6. In Figure 6.6, $\rho$ denotes the radius of the path $PQ$ at $P$, $\hat{W}$ is the unit tangent at $P$, and $\hat{V}$ the unit normal. The displacement from $P$ to $Q$ can be expressed in the form $u = \xi \hat{W} + \eta \hat{V}$ or, if $\epsilon$ denotes arc length along $PQ$,

$$u = \epsilon \frac{du}{d\epsilon} + \frac{1}{2} \epsilon^2 \frac{d^2 u}{d\epsilon^2} + \cdots = \epsilon \hat{W} + \frac{\epsilon^2}{2 \rho} \hat{V} + \cdots.$$  

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If we accept an approximation of second degree in the arc length $\epsilon$, then

$$\xi = \epsilon, \quad \eta = \frac{\epsilon^2}{2 \rho}.$$  \hspace{1cm} (6.44a, b)

In the $N$-dimensional space, one can define an arc length $\epsilon$ along a path stemming from the critical state; that is,

$$du_i du_i = d\epsilon^2.$$

A component of the unit tangent is

$$\overline{W_i} = \frac{du_i}{d\epsilon}.$$ \hspace{1cm} (6.45)

A component of the unit normal is

$$\kappa V_i = \frac{d^2 u_i}{d\epsilon^2}.$$ \hspace{1cm} (6.46)

The displacement along a small segment is

$$u_i = \frac{du_i}{d\epsilon} + \frac{1}{2} \frac{d^2 u_i}{d\epsilon^2} \epsilon^2 + \cdots$$

$$= \epsilon \overline{W_i} + \frac{\epsilon^2}{2} \kappa V_i + \cdots.$$ \hspace{1cm} (6.47)

Here, $V_i$ is normalized in the manner of (6.43). In essence, we require the normal $V_i$ and curvature $\kappa$ which determine the curved path of minimum change $\Delta V$. The change of potential follows from (6.33) and (6.47) and simplifies according to (6.35), (6.38), and (6.40):

$$\Delta V = \frac{\epsilon^3}{3!} A_{ijk} \overline{W_i} \overline{W_j} \overline{W_k} + \frac{\epsilon^4}{8} \kappa^2 A_{ij} V_i V_j$$

$$+ \frac{\epsilon^4}{4!} A_{ijkl} \overline{W_l} \overline{W_j} \overline{W_k} \overline{W_i} + \frac{\epsilon^4}{4} \kappa A_{ijk} \overline{W_i} \overline{W_j} V_k + O(\epsilon^5).$$ \hspace{1cm} (6.48)

If $\epsilon$ is sufficiently small, the initial term of (6.48) is dominant. Since the sign of the initial (cubic) term can be positive or negative, depending on
the sense of the displacement $\bar{W}_i$, a necessary condition for stability follows:

$$A_3 \equiv \frac{1}{3!} A_{ijk} \bar{W}_i \bar{W}_j \bar{W}_k = 0. \quad (6.49)$$

If $A_3$ vanishes, as it usually does in the case of a symmetric structure, then the sign of $\Delta V$ rests with the terms of higher degree. If $\Delta V$ is negative for one displacement $V_i$, then the system is unstable. The minimum of (6.48) is stationary, that is, $\delta(\Delta V) = 0$, for variations of $V_i$. The stationary conditions follow

$$A_{ij} \kappa V_j = -A_{jki} \bar{W}_j \bar{W}_k V_i + \cdots + O(\epsilon). \quad (6.50)$$

If the terms of higher degree are neglected, then equation (6.50) constitutes a linear system in the displacement $V_j$. In accordance with (6.45) and (6.46), the solution $\bar{V}_i$ is to satisfy the orthogonality condition:

$$\bar{V}_i \bar{W}_i = 0. \quad (6.51)$$

It follows from (6.50) that

$$\kappa^2 A_{ij} \bar{V}_i \bar{V}_j = -\kappa A_{jki} \bar{W}_j \bar{W}_k \bar{V}_i + \cdots + O(\epsilon). \quad (6.52)$$

The potential change corresponding to the displacement

$$u_i = \epsilon \bar{W}_i + \frac{\epsilon^2}{2} \kappa \bar{V}_i$$

is obtained from (6.48) and simplified by means of (6.49) and (6.52):

$$\Delta V = \epsilon^4 A_4, \quad (6.53)$$

where

$$A_4 \equiv \frac{1}{4!} A_{ijkl} \bar{W}_i \bar{W}_j \bar{W}_k \bar{W}_l - \frac{\kappa^2}{8} A_{ij} \bar{V}_i \bar{V}_j. \quad (6.54)$$

The system is stable if

$$A_4 > 0. \quad (6.55a)$$

The system is unstable if

$$A_4 < 0. \quad (6.55b)$$
In a system with one degree of freedom, $V_i = 0$, and the final term of (6.54) vanishes.

### 6.12 Equilibrium States Near the Critical Load

In our preceding view of stability at the critical load $\lambda^*$, we examined the energy increment upon excursions from the critical state but assumed that the load remained constant. Such excursions follow the path of minimum potential on a hyperplane ($\lambda = \lambda^*$) in the configuration-load space $(q_i; \lambda)$. To trace a path of equilibrium from the critical state requires, in general, a change in the load. Let us now explore states of equilibrium near the reference state of equilibrium $(q_i; \lambda^*)$.

To this end, we assume that the potential $V(q_i; \lambda)$ can be expanded in a Taylor’s series in the load $\lambda$, as well as the displacement $u_i$. Then, in place of (6.33), we have

\[
\Delta V = \left( \bar{A}_i u_i + \frac{1}{2} A_{ij} u_i u_j + \frac{1}{3!} A_{ijk} u_i u_j u_k + \cdots \right) + \left( \bar{A}'_i u_i + \frac{1}{2} A'_{ij} u_i u_j + \frac{1}{3!} A'_{ijk} u_i u_j u_k + \cdots \right) (\lambda - \lambda^*) + \cdots. \tag{6.56}
\]

Here, the prime (′) signifies the derivative with respect to the parameter $\lambda$. Note that each of the coefficients ($\bar{A}_i, \bar{A}'_i$, etc.) is evaluated at the critical load.

Along a smooth path from the reference state in the configuration-load space, the “path” includes a step in the direction of $\lambda$, as well as the direction of $q_i$. In place of (6.47), we have

\[
u_i = \epsilon u'_i + \frac{\epsilon^2}{2} \kappa V_i + \cdots, \tag{6.57a}
\]

\[
(\lambda - \lambda^*) = \epsilon \lambda' + \frac{\epsilon^2}{2} \kappa \mu + \cdots. \tag{6.57b}
\]

Here, the vector $(u'_i; \lambda')$ is the unit tangent and $(V_i; \mu)$ is the principal normal at $(q_i; \lambda^*)$ of the path which traces equilibrium states in the space of configuration-load $(q_i; \lambda)$.

---

‡The logic follows the thesis of W. T. Koiter [107].
Upon substituting (6.57a, b) into (6.56), we obtain

$$\Delta V = \epsilon\bar{A}_i u'_i + \epsilon^2\left(\frac{1}{2}A_{ij}u'_iu'_j + \bar{A}'_i u'_i \lambda' + \frac{1}{2}\kappa\bar{A}_i V_i \right) + \cdots.$$  \hspace{1cm} (6.58)

The principle of stationary potential energy gives the equations of equilibrium at the reference state:

$$\bar{A}_i = 0.$$  \hspace{1cm} (6.59)

In view of (6.59), the quadratic terms ($\epsilon^2$) dominate (6.58). The stationary principle, $\delta(\Delta V) = 0$, gives the equilibrium equations for states very near the reference state:

$$A_{ij}u'_j = -\bar{A}'_i \lambda'.$$  \hspace{1cm} (6.60)

Now, the reference state is critical if

$$\bar{A}'_i \lambda' = 0.$$  \hspace{1cm} (6.61)

In words, either $\lambda' = 0$, which implies the existence of an adjacent state at the same level of loading, and/or $\bar{A}'_i = 0$; the latter holds if the reference configuration is an equilibrium configuration for $\lambda \neq \lambda^*$. Then, the equilibrium equations of the neighboring state follow:

$$A_{ij}u'_j = 0.$$  \hspace{1cm} (6.62)

Equations (6.62) are the equations (6.40) of the Trefftz condition (6.39). The solution of (6.62) is the buckling mode

$$u'_i = \bar{W}_i.$$  \hspace{1cm} (6.63)

Suppose that $\bar{A}'_i = 0$ in (6.61) and $\lambda' \neq 0$. Then, according to (6.59), (6.62), and (6.63), the potential of (6.56) and (6.58) takes the form:

$$\Delta V = \epsilon^3(A_3 + A'_2 \lambda' + \cdots) + O(\epsilon^4),$$  \hspace{1cm} (6.64)

where

$$A_3 \equiv \frac{1}{3!}A_{ijk}\bar{W}_i\bar{W}_j\bar{W}_k,$$  \hspace{1cm} (6.65)

$$A'_2 \equiv \frac{1}{2}A'_{ij}\bar{W}_i\bar{W}_j.$$  \hspace{1cm} (6.66)
We accept the indicated terms of (6.64) as our approximation and, in accordance with (6.57b), set

\[ \epsilon' = \lambda - \lambda^*. \]  

(6.67)

Our approximation of (6.64) follows:

\[ \Delta V' = \epsilon^2 A_3 + \epsilon^2 A_2' (\lambda - \lambda^*). \]  

(6.68)

The principle of stationary potential provides the equation of equilibrium:

\[ \frac{d \Delta V}{d \epsilon} = 3\epsilon A_3 + 2\epsilon A_2' (\lambda - \lambda^*) = 0, \]  

(6.69a)

or

\[ \epsilon = -\frac{2A_2'}{3A_3} (\lambda - \lambda^*). \]  

(6.69b)

The state is stable if the potential is a minimum, that is, if

\[ \frac{d^2 \Delta V}{d \epsilon^2} = 6\epsilon A_3 + 2\epsilon A_2' (\lambda - \lambda^*) > 0, \]  

(6.70a)

or, in accordance with (6.69b), the system is stable in the adjacent state if

\[ -A_2' (\lambda - \lambda^*) > 0, \quad A_2' (\lambda - \lambda^*) < 0. \]  

(6.70b, c)

In accordance with (6.56), (6.62), and (6.63), the quadratic terms of \( \Delta V \) in the buckled mode follow:

\[ V_2(W_i) \equiv \frac{1}{2} \left[ A_{ij} + A_{ij}' (\lambda - \lambda^*) + \frac{1}{2} A_{ij}'' (\lambda - \lambda^*)^2 \right] W_i W_j. \]

Since \( V_2(W_i) = 0 \) at the critical load, we expect that \( V_2(W_i) > 0 \) at loads slightly less than the critical value and that \( V_2(W_i) < 0 \) at loads slightly above the critical value. Therefore, we conclude that

\[ A_2' \equiv \frac{1}{2} A_{ij}' W_i W_j < 0. \]  

(6.71)

According to (6.71), the numerator of (6.69b) is always negative, but the denominator of (6.69b) is a homogeneous cubic in \( W_i \) and the sign is
reversed by a reversal of the buckling mode. In this case, an adjacent state of equilibrium exists at loads above \((\lambda > \lambda^*)\) or below \((\lambda < \lambda^*)\) the critical value. In view of (6.70b) and (6.71), an equilibrium state above the critical load is stable and a state below is unstable.

Now, suppose that

\[
\lambda' = A_3 = 0, \quad (6.72)
\]

then, in view of (6.59), (6.62), (6.63), and (6.72), the potential of (6.56) and (6.58) takes the form:

\[
\Delta V = \epsilon^4 \left( \frac{\kappa^2}{8} A_{ij} V_i V_j + \frac{\kappa}{4} A_{ijk} V_i W_j W_k + \frac{1}{4!} A_{ijkl} W_i W_j W_k W_l + \cdots \right) + O(\epsilon^5). \quad (6.73)
\]

The underlined term of (6.73) dominates if \(\mu \neq 0\) and if \(\epsilon\) is sufficiently small. The term is odd in \(W_i\) and, therefore, always provides a negative potential change at any load \(\lambda \neq \lambda^*\). A condition for the existence of stable states at noncritical values of load follows:

\[
\bar{A}'_i W_i = 0. \quad (6.74)
\]

However, the buckling mode \(W_i\) is independent of the coefficients \(\bar{A}'_i\). Therefore, equation (6.74) implies generally that

\[
\bar{A}'_i = 0. \quad (6.75)
\]

Now, we accept the remaining terms indicated in (6.73) as our approximation. Also, in view of (6.57b) and (6.72),

\[
\epsilon^2 \frac{\kappa}{2} \mu \doteq \lambda - \lambda^*. \quad (6.76)
\]

Our approximation of (6.73) follows:

\[
\Delta V \doteq \epsilon^4 \left( \frac{\kappa^2}{8} A_{ij} V_i V_j + \frac{\kappa}{4} A_{ijk} V_i W_j W_k + \frac{1}{4!} A_{ijkl} W_i W_j W_k W_l \right) + \left( \epsilon^2 \frac{1}{2} A'_{ij} W_i W_j \right) (\lambda - \lambda^*). \quad (6.77)
\]
Again, we require a stationary potential for variations of the displacement \(V_i\). The equations of equilibrium follow:

\[
\epsilon^2 \kappa A_{ij} V_j = -\epsilon^2 A_{ijk} W_j W_k. \tag{6.78}
\]

Let \(\kappa V_i\) denote the solution of (6.78). Then, it follows that

\[
\epsilon^2 \kappa^2 A_{ij} V_i V_j = -\epsilon^2 \kappa A_{ijk} W_j W_k V_i. \tag{6.79}
\]

If the solution \(\kappa V_i\) and (6.79) are used in (6.77), then our approximation of the potential takes the form:

\[
\Delta V = \epsilon^4 A_4 + \epsilon^2 (\lambda - \lambda^*) A'_2, \tag{6.80}
\]

where \(A'_2\) is defined by (6.66) and \(A_4\) by (6.54).

The solution of (6.78) determines the unit vector \(V_i\) which renders \(\Delta V\) stationary, but still dependent upon the distance \(\epsilon\). The principle of stationary potential gives the equilibrium condition:

\[
\frac{d \Delta V}{d \epsilon} = 4\epsilon^3 A_4 + 2\epsilon A'_2 (\lambda - \lambda^*) = 0, \tag{6.81}
\]

or

\[
\epsilon^2 = -\frac{A'_2}{2A_4} (\lambda - \lambda^*). \tag{6.82}
\]

According to (6.55a,b), (6.71), and (6.82), a stable adjacent state of equilibrium can exist only at loads above the critical value \((\lambda > \lambda^*)\) and a state below the critical value is unstable.

W. T. Koiter [107] provides rigorous arguments for the conditions (6.71) and (6.75) if the critical configuration is a stable equilibrium configuration for loads less than the critical value. For example, the two-dimensional system has equilibrium configurations which trace a line along the \(\lambda\) axis, as shown in Figure 6.7. The portion \(OP\) represents stable states, the bifurcation point \(P\) represents the critical state, \(PR\) represents unstable states of the reference configuration, and \(PQ\) represents stable postbuckled equilibrium states. Here, the principle of stationary energy in the critical configuration at \(any\) load leads to equation (6.75), and the principle of minimum energy in the stable states of \(OP\) \((\lambda < \lambda^*)\) leads to the inequality (6.71).

If the cubic term of \(\Delta V\) does not vanish, then equilibrium states trace paths with slope \(\lambda'\) at the critical load, as shown in Figure 6.7a. If the
cubic term vanishes, then $\lambda' = 0$, and the equilibrium states trace paths as shown in Figure 6.7b. In each figure, the solid lines are stable branches and the dotted lines are unstable.

Practically speaking, many structural systems display the instability patterns of Figure 6.7, that is, the prebuckled configuration of the ideal structure is an equilibrium state under all loads. Notable examples are the column under axial thrust, the spherical or cylindrical shell under external pressure, and the cylinder under uniform axial compression. Theoretically, each retains its form until the load reaches the critical value and then buckles. In the case of thin shells, initial imperfections cause pronounced departures from the initial form and often cause premature buckling ($\lambda \ll \lambda^*$).

Our analysis of stability at the critical load is limited. The reader should note, especially, that any of the various terms of the potential, for example, $V_2, V_4$, may vanish identically. Then, further investigation, involving terms of higher degree, is needed.

6.13 Effect of Small Imperfections upon the Buckling Load

In the monumental work of W. T. Koiter [107], an important practical achievement was his assessment of the effect of geometric imperfections upon the buckling load of an actual structure. Here, we outline the proce-
Under the conditions of dead loading upon the Hookean structure, the energy potential $\Delta \tilde{V}$ of the actual structure is expressed in terms of a displacement $u_i$ from the critical state of the ideal structure and a parameter $e$ which measures the magnitude of the initial displacement of the actual unloaded structure:

$$\Delta \tilde{V} = \left( \frac{1}{2} A_{ij} u_i u_j + \frac{1}{3!} A_{ijk} u_i u_j u_k + \frac{1}{4!} A_{ijkl} u_i u_j u_k u_l + \cdots \right)$$

$$+ \left( \frac{1}{2} A'_{ij} u_i u_j + \frac{1}{3!} A'_{ijk} u_i u_j u_k + \cdots \right) (\lambda - \lambda^*)$$

$$+ e \left[ B_i u_i + \frac{1}{2} B_{ij} u_i u_j + \cdots \right] + \cdots. \quad (6.83)$$

Here, the linear terms in $u_i, \bar{A}_i, \text{and} \bar{A'}_i$ vanish because the reference configuration is an equilibrium configuration of the ideal structure ($e = 0$) at any load. Note: For all purposes, the coefficients $(A_{ij}, \ldots, B_{ij}, \ldots)$ are entirely symmetric with respect to the indices.

As before, the components $u_i$ are expanded in powers of the arc length $\epsilon$ along the ideal curve of Figure 6.8. Here, we make an assumption that the initial deflection of the actual structure is nearly the buckling mode $\bar{W}_i$ of the ideal structure. Therefore, we have

$$u_i = \epsilon \bar{W}_i + \epsilon^2 \frac{k}{2} V_i. \quad (6.84)$$

Here, the second-order term of (6.84) contains an unspecified parameter $k$, because this term does not represent a deviation from the tangent ($\epsilon \bar{W}$) along the ideal path of Figure 6.8, but represents the displacement $d$ which carries the system to the actual path as depicted in Figure 6.8.

Upon substituting (6.84) into (6.83) and acknowledging (6.38), (6.40), and (6.42), we obtain

$$\Delta \tilde{V} = \epsilon^3 A_3 + \epsilon^2 A'_2 (\lambda - \lambda^*) + O(\epsilon^2)(\lambda - \lambda^*)^2 + O(\epsilon^5)$$

$$+ \epsilon^4 \left[ \frac{1}{4!} A_{ijkl} \bar{W}_i \bar{W}_j \bar{W}_k \bar{W}_l + \frac{k^2}{8} A_{ij} V_i V_j + \frac{k}{4} A_{ijk} \bar{W}_i \bar{W}_j V_k \right]$$

$$+ e \left[ B_i \bar{W}_i + \epsilon^2 B_{ij} \frac{k}{2} V_i \right], \quad (6.85)$$

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where $A_3$ and $A'_2$ are defined by (6.65) and (6.66), as before. Since the relative magnitudes of $\epsilon$ and $e$ are unspecified, we must suppose that the terms $O(\epsilon^3)$ and $O(\epsilon e)$ dominate (6.85), if $A_3 \neq 0$ and $\lambda - \lambda^* \neq 0$. Then, we have the approximation

$$\Delta \tilde{V} \simeq \epsilon^3 A_3 + \epsilon^2 A'_2 (\lambda - \lambda^*) + \epsilon e B_1 W_i. \quad (6.86)$$

The stationary condition of equilibrium follows:

$$\frac{d \Delta \tilde{V}}{d\epsilon} = 3\epsilon^2 A_3 + 2\epsilon A'_2 (\lambda - \lambda^*) + \epsilon e B_1 W_i = 0. \quad (6.87)$$

Now, recall that $\Delta \tilde{V}$ is not the potential increment from the critical configuration of the actual structure but the potential referred arbitrarily to the critical configuration of the ideal structure. An equilibrium configuration of the actual structure is stable or unstable, respectively, if the potential is a minimum or a maximum; therefore, the critical load $\bar{\lambda}$ of the actual
structure satisfies the conditions

$$\frac{d^2 \Delta \hat{V}}{d\epsilon^2} = 6\epsilon A_3 + 2A'_2(\lambda - \lambda^*), \quad (6.88)$$

$$> 0 \quad \Rightarrow \quad \text{stability}, \quad (6.89a)$$

$$< 0 \quad \Rightarrow \quad \text{instability}. \quad (6.89b)$$

Observe that the distinction between stability and instability of a postbuckled state rests upon the same conditions, (6.89a, b), as the ideal structure [see equation (6.70a)] and that the conditions are independent of the imperfection parameter \( \epsilon \).

If \( A_3 \neq 0 \), then the imperfect structure deflects and reaches a critical state of equilibrium when (6.87) is satisfied and (6.88) vanishes. If \((\lambda - \lambda^*)\) is eliminated from the two equations, then

$$e = \frac{3\epsilon^2 A_3}{B_i \overline{W_i}}. \quad (6.90)$$

The sign of the sum \( B_i \overline{W_i} \) is arbitrary, since a change of sign is effected by redefining the parameter \( e \). Therefore, we can choose \( e \) so that \( B_i \overline{W_i} \) has the opposite sign of \( A_3 \). Then the condition (6.90) for a critical state is fulfilled only if \( e < 0 \). From our observations, we know that an imperfect structure tends to buckle in a preferred direction, depending upon the character of the geometric deviations. In the present case, if \( A_3 < 0 \), a critical state occurs only if \( e < 0 \). A plot of load versus deflection is depicted in Figure 6.9a; here, a negative value \( e \) produces buckling in a negative mode \((\epsilon \overline{W_i} < 0)\) according to the curve \( O\hat{P} \), whereas the positive value \( e \) produces only stable states along the path \( OP \).

A real structure which behaves in the manner of Figure 6.9a is the frame of Figure 6.9b. If the vertical strut is bent to the right or left, the imperfection parameter \( e \) is negative or positive, respectively. The rotation \( \theta \) of the joint serves as a generalized coordinate \((\theta = q)\) and plots of load versus rotation take the forms of Figure 6.9a. The frame under eccentric loading has been studied experimentally by J. Roorda [111] and theoretically by W. T. Koiter [112], [113]. The latter computations show remarkable agreement with the former experimental results.

Let us turn to a structure that exhibits the behavior of Figure 6.7b, characterized by the condition

$$A_3 = 0. \quad (6.91)$$

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At the critical load, terms $O(\epsilon^3)$ are absent from the potential of (6.85). The latter must be stationary with respect to the displacement $V_i$; for equilibrium,

$$
\epsilon^2 k A_{ij} V_j = -\epsilon^2 A_{jkl} W_j W_k V_l - 2\epsilon B_i.
$$

(6.92)

In this instance, we pursue a path (from $P$ in Figure 6.7b) in the space $(q_i, \lambda)$ with $\lambda = \lambda^*$. If $kV_i$ denotes the solution of (6.92), then

$$
\epsilon^2 k A_{jkl} W_j W_k V_i = -\epsilon^2 k^2 A_{ij} V_i V_j - 2\epsilon B_i V_i.
$$

(6.93)

In accordance with (6.91) and (6.93), our approximation of (6.85) follows:

$$
\Delta \tilde{V} \doteq \epsilon^4 A_4 + \epsilon^2 A'_2 (\lambda - \lambda^*) + \epsilon \epsilon B_i W_i.
$$

(6.94)

The coefficient $A'_2$ is defined by (6.66) and $A_4$ is defined, much as $A_4$ in (6.54):

$$
A_4 \equiv 1 \frac{1}{q} A_{ijkl} W_i W_j W_k W_l - \frac{k^2}{8} A_{ij} V_i V_j,
$$

where $k$ is the parameter in (6.84). As before, the potential $\Delta \tilde{V}$ is still dependent upon the distance $\epsilon$. The stationary condition of equilibrium
follows:
\[
\frac{d \Delta \tilde{V}}{d \epsilon} = 4\epsilon^3 \tilde{A}_4 + 2\epsilon A_2' (\lambda - \lambda^*) + \epsilon B_i \bar{W}_i = 0. \tag{6.95}
\]

The stability of equilibrium depends upon the second derivative as follows:
\[
\frac{d^2 \Delta \tilde{V}}{d \epsilon^2} = 12\epsilon^2 \tilde{A}_4 + 2A_2' (\lambda - \lambda^*), \tag{6.96}
\]
\[
> 0 \implies \text{stability}, \tag{6.97a} \]
\[
< 0 \implies \text{instability}. \tag{6.97b}
\]

Again, the critical state of equilibrium is characterized by vanishing of the first derivative (6.95) and second derivative (6.96). The elimination of \((\lambda - \lambda^*)\) yields the result:
\[
e = \frac{8\epsilon^3 \tilde{A}_4}{B_i \bar{W}_i}. \tag{6.98}
\]

Again, we note that the definition of \(e\) and the sign of \(\bar{W}_i\) are arbitrary and, therefore, we assume that \(\bar{W}_i\) renders the sum \(B_i \bar{W}_i > 0\). Then, the

---

Figure 6.10 Effect of imperfections on a symmetrical system

\[Q\]
\[\tilde{Q}\]
\[\tilde{\lambda}\]
\[\lambda^*\]
\[\lambda\]
\[\bar{A}_4 > 0\]
\[\bar{A}_4 < 0\]
\[\bar{A}_4 > 0\]
\[\bar{A}_4 < 0\]
\[e > 0\]
\[e < 0\]
\[q\]
condition (6.98) for a critical load is attained if $\bar{A}_4 < 0$, $e < 0$, in keeping with (6.55b). Now, the structure also exhibits instability at the critical load if the sign of the parameter $e$ and the buckling mode are both reversed. A plot of load versus deflection is depicted in Figure 6.10; it is characteristic of a symmetrical structure (e.g., a column) which could buckle in either direction depending upon the inherent imperfections. In either case, it is stable or unstable depending upon the sign of the constant $\bar{A}_4$. The critical load $\bar{\lambda}$ of the actual structure may be much less than the critical load $\lambda^*$ of the ideal structure.

The instability of the imperfect systems are governed by equations (6.87) and (6.88), or by equations (6.95) and (6.96). The former prevail when $e < 0$ and $A_3 < 0$ (see Figure 6.9a). The latter govern when $A_3 = 0$ and $\bar{A}_4 < 0$ (see Figure 6.10). The effect of the imperfection $e$ is assessed by eliminating the parameter $\epsilon$ from (6.87) and (6.88) in the first instance, or from (6.95) and (6.96) in the second. In the first instance ($A_3 < 0$), the critical load $\bar{\lambda} < \lambda^*$ is thereby expressed in terms of the imperfection ($-\epsilon$):

$$\bar{\lambda} = \lambda^* - 3(-\epsilon B_i W_i)(-A_3)^{1/2}(-A'_2)^{-1}. \quad (6.99)$$

In the second instance ($A_3 = 0$, $\bar{A}_4 < 0$), the corresponding relation between $\bar{\lambda}$ and $\lambda^*$ assumes the following form:

$$\bar{\lambda} = \lambda^* - \frac{3}{2}(-\epsilon B_i W_i)^2(-\bar{A}_4)^{1/3}(-A'_2)^{-1}. \quad (6.100)$$

Plots of the actual buckling load versus the imperfection parameter for both cases are presented in Figure 6.11. Note that small imperfections can cause
considerable reduction of the buckling load. Especially, the system $A_3 < 0$, as in Figure 6.9 is more susceptible to the imperfection.

6.14 Principle of Virtual Work Applied to a Continuous Body

The concepts of virtual displacement and work, and the principle as previously stated in Section 6.5, hold for any mechanical system. These apply as well to a continuous body; only the mathematical formulations differ. The continuum of particles undergoes a continuously variable displacement $\mathbf{V}$, and also the virtual displacement $\delta\mathbf{V}$. Hence, the finite number of discrete variables $q_i$ are replaced by a continuous function $\mathbf{V}$. Continuous internal and external forces replace the discrete forces $'F^i$ and $P^n$. The former are expressed by a stress $s^i$ (associated with the initial area), the latter by external body force $f$ (per unit initial volume) and surface traction $T$ (per unit initial area). Now, the virtual work is expressed by integrals extending over the initial volumes and surfaces. Note: The current equilibrium state is a deformed state; the stresses and forces act upon deformed surfaces and volumes. Our choices of measurement (initial volume and surfaces) are pragmatic: In practice, one knows only the initial-reference state, and those dimensions; one seeks the deformed-current state.

The virtual work of the stresses has any of the forms given previously in Section 4.5; there, increments are signified by the overdot ($\cdot$). Here, the increments of displacement and corresponding strain are the arbitrarily imposed virtual increments signified by $\delta\mathbf{V}$ and $\delta\gamma_{ij}$. The virtual work expended by stresses (per unit volume) follows [see (4.19c–e) and (4.21b, c)]:

$$\delta w_s = \sqrt{g^{ii}} \sigma^i \cdot \delta\mathbf{V},_i = s^i \cdot \delta G_i, \quad (6.101a)$$

$$= s^{ij} G_j \cdot \delta G_i, \quad (6.101b)$$

$$= t^{ij} \dot{g}_j \cdot \delta G_i. \quad (6.101c)$$

Notice that we give alternative representations: One incorporates tensorial components $s^{ij}$ associated with the current/deformed vectors $\mathbf{G}_i$, the other embodies components $t^{ij}$ associated with the rigidly rotated, but undeformed vectors $\dot{g}_i$.

Let us note at the onset that our virtual displacements and, subsequently, variations of displacement are infinitesimals (small of first order); addi-
tionally, the displacement and derivatives are presumed continuous. From equations (3.114) and (3.115a), it follows that

$$\delta V = \delta R, \quad \delta R_i = \delta G_i = \delta V_i.$$  

The work we must expend against the body force (per unit volume) is

$$\delta w_f = -f \cdot \delta V = -f \cdot \delta R.$$ \hspace{1cm} (6.102)

We suppose that tractions $T$ act upon a part $s_t$ of the bounding surface $s$. The virtual work that we do (per unit area) is

$$\delta w_t = -T \cdot \delta V = -T \cdot \delta R.$$ \hspace{1cm} (6.103)

Finally, the virtual work $\delta W$ upon the entire body is comprised of the integrals of $\delta w_s$ and $\delta w_f$ throughout the volume $v$, and $\delta w_t$ over the surface $s_t$:

$$\delta W = \delta W_s + \delta W_f + \delta W_t,$$ \hspace{1cm} (6.104a)

$$\delta W = \iiint_v (s^i \cdot \delta G_i - f \cdot \delta V) \, dv - \iint_{s_t} T \cdot \delta V \, ds.$$ \hspace{1cm} (6.104b)

Here, according to (6.101b, c), we have the alternative forms of the vector $s^i$:

$$s^i = s^{ij} G_j = t^{ij} \dot{g}_j,$$ \hspace{1cm} (6.105a, b)

We note that components of the increments $\delta G_i$ can be treated, mathematically, as the rates of (3.139) and (3.147) (Section 3.18), viz.,

$$\delta G_i = (\delta \gamma_{ji} + \delta \omega_{ji}) G^j,$$ \hspace{1cm} (6.106a)

$$= \delta h_{ij} \dot{g}^j + C_{ij} \delta \omega \times \dot{g}^j = (\delta h_{ij} + C_{ij} \delta \varpi^{kl} e_{klj}) \dot{g}^j.$$ \hspace{1cm} (6.106b, c)

In each form, the first term comprises the deformation; moreover, both strains are symmetric,

$$\delta \gamma_{ij} = \delta \gamma_{ji}, \quad \delta h_{ij} = \delta h_{ji}.$$  

In each, the final term represents solely the rigid rotation. The virtual work of the stresses consists of the two parts, but only the symmetrical parts
\((s^{ij} = s^{ji} = s^{ji} = s^{ij})\) do work upon the respective strains. Therefore,

\[
\delta w_s = s^{ij} \delta \gamma_{ji} + s^{ij} \delta \omega_{ji} = t^{ij} \delta h_{ij} + t^{ij} (C_i^l \delta \overline{\epsilon}_{klj}). \tag{6.107a, b}
\]

Equilibrium of an arbitrarily small element requires that the virtual work vanish for arbitrary virtual displacements. In this case, the rigid motions \((\delta V\) and \(\delta \omega)\) and the deformation \((\delta \gamma_{ij} or \delta h_{ij})\) must be regarded as independent. Hence, equilibrium requires that the final term of (6.107a, b) vanish:

\[
s^{ij} \delta \omega_{ji} = 0, \quad t^{ij} (C_i^l \delta \overline{\epsilon}_{klj}) = 0. \tag{6.108a, b}
\]

These terms must vanish for arbitrary rotation; since \(\delta \omega_{ji} = -\delta \omega_{ij}\) and \(e_{klj} = -e_{kjl}\),

\[
s^{ij} = s^{ji}, \quad t^{ij} C_i^l = t^{il} C_j^l. \tag{6.109a, b}
\]

Then the expression (6.104) incorporates only the virtual work of the symmetric tensor \(s^{ij}\) or the symmetric part \(t^{ij}\) of the stress \(s^i\).

Having established the equilibrium conditions (6.109a, b), we return to the virtual work (6.104), but recall that \(\delta V = \delta R\) and \(\delta G_i = \delta R_i\), with the requisite continuity of the vector \(R\), derivatives and variations:

\[
\delta W = \iiint_v (s^i \cdot \delta R_i - f \cdot \delta R) \sqrt{g} \, d\theta_1 \, d\theta_2 \, d\theta_3 - \iint_{s_i} (T - s^i n_i) \cdot \delta R \, ds. \tag{6.110a}
\]

Applying Green’s theorem to the first term [see equation (2.73)], we obtain

\[
\delta W = -\iiint_v \left[ \frac{1}{\sqrt{g}} (\sqrt{g} s^i)_i + f \right] \cdot \delta R \, dv - \iint_{s_v} (T - s^i n_i) \cdot \delta R \, ds + \iint_{s_v} (s^i n_i) \cdot \delta R \, ds. \tag{6.110b}
\]

Here, \(n_i = \hat{n} \cdot g_i\) is the component of the initial normal to the surface \(s\); \(s_v = s - s_t\) is that part of the bounding surface \(s\) where displacements are prescribed.

The principle of virtual work requires that the virtual work vanishes \((\delta W = 0)\) for arbitrary admissible virtual displacements \(\delta R\), so-called variations (see e.g., [10], [29]). Admissibility implies that the variation \(\delta R\) has
the requisite continuity and is consistent with the constraints ($\delta R$ vanishes where displacements are prescribed, e.g., where the boundary surface is fixed). The condition is fulfilled if, and only if,

$$\frac{1}{\sqrt{g}}(\sqrt{g} s^i)_{,i} + f = 0 \quad \text{in} \quad v, \quad (6.111)$$

$$s^i n_i = T \quad \text{on} \quad s_t, \quad (6.112)$$

$$\delta R = 0 \quad \text{on} \quad s_v. \quad (6.113)$$

The first equation (6.111) is the differential equation of equilibrium (4.44b) which can be expressed in terms of any of the stress tensors ($s^{ij}$ or $t^{ij}$); see (4.45b, c). Equation (6.112) asserts the equilibrium of internal actions $s^i n_i$ and surface traction $T$ upon an element at surface $s_t$. The last equation (6.113) expresses the fact that the displacements are prescribed on $s_v$.

---

**6.15 Principle of Stationary Potential Applied to a Continuous Body**

If the internal forces are conservative (practically speaking, if the body is elastic), then their incremental work derives from a potential:

$$\delta w_a = -\delta u_0 = -s^{ij} \delta \gamma_{ij}, \quad (6.114a)$$

$$\delta \bar{u}_0 = -t^{ij} \delta h_{ij}. \quad (6.114b)$$

The function $u_0 = u_0(\gamma_{ij})$ [or $\bar{u}_0(h_{ij})$] may be the internal energy or the free energy (see Section 5.11) when the deformation is adiabatic or isothermal, respectively. In either case,

$$s^{ij} = \frac{\partial u_0}{\partial \gamma_{ij}}, \quad t^{ij} = \frac{\partial \bar{u}_0}{\partial h_{ij}}. \quad (6.115a, b)$$

The incremental work of all internal forces is the variation of an internal-
energy potential $U^{\dagger}$

$$\delta W_s = -\delta U \equiv - \iiint_v \delta u_0 \, dv. \quad (6.116)$$

If the body forces are conservative, the incremental work of body forces is the negative of the variation of a potential $\Pi_f$:

$$\delta W_f = -\delta \Pi_f \equiv - \iiint_v \delta \pi_f(X_i) \, dv, \quad (6.117)$$

where $\pi_f$ is the potential (per unit initial volume) and $X_i$ is the current Cartesian coordinate of the particle ($X_i \equiv R \cdot \hat{i}_i$). Since $X_i = \hat{i}_i \cdot R(\theta^i)$, the potential is also implicitly a function of the arbitrary coordinate $\theta^i$. The body force has alternative forms:

$$f = -\frac{\partial \pi_f}{\partial X_i} \hat{i}_i = f^i g_i = f^i G_i. \quad (6.118)$$

Likewise, if the surface tractions are conservative, the incremental work of surface forces is the negative of the variation of a potential $\Pi_t$:

$$\delta W_t = -\delta \Pi_t \equiv - \iint_{s_{\Sigma}} \delta \pi_t(X_i) \, ds. \quad (6.119)$$

The traction also has alternative representations:

$$T = -\frac{\partial \pi_t}{\partial X_i} \hat{i}_i = T^i g_i = T^i G_i. \quad (6.120)$$

The principle of virtual work of (6.104a, b) applies to the conservative system, wherein

$$\delta W = -\delta \mathcal{V} \equiv -(\delta U + \delta \Pi_f + \delta \Pi_t) = 0. \quad (6.121)$$

In words, the system is in equilibrium if, and only if, the first-order variation $\delta \mathcal{V}$ of the total potential $\mathcal{V}^{\dagger}$ vanishes; or stated otherwise, the potential is

$^{\dagger}$Note that the integral $U = \iiint_v u \, dv$ has a value for each function $u(X_i)$; such quantities are called functionals. If $u$ is assigned a variation $\delta u(X_i)$, the corresponding change in $U$ is termed the variation of $U$ and is denoted by $\delta U$. Note also that $\delta W_s$ is not, in general, the variation of a functional.

$^{\dagger}$The first-order variation, or, simply, first variation, signifies a variation which includes only terms of first order in the varied quantity, in this instance, the displacement.

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stationary. Again, as in (6.104a, b) and (6.110a, b), the variation is arbitrary but subject to constraints (e.g., $V = \nabla = \mathbf{0}$ on $s_v$) and requisite continuity. The stationary conditions are again the equilibrium conditions (6.109a, b) (arbitrary rotation), equation (6.111) (arbitrary displacement in $v$), and equation (6.112) (arbitrary displacement on $s_t$). Finally, we have also the constraint (6.113) ($\delta \mathbf{R} = \mathbf{0}$ on $s_v$).

The energy criteria for stability of equilibrium are presented in Sections 6.10 through 6.13. Those criteria rest upon the changes of potential which accompany small movements about a state of equilibrium. The analyses provide means to determine the critical loads which signal buckling of the perfect system, but also certain consequences of imperfections. The buckling can be the gradual, but excessive, deflection of a simple column or the abrupt severe disfiguration of a shell, so-called snap-buckling. The latter are also the most susceptible to imperfections and most prone to premature buckling. These effects in the continuous systems are analyzed as the similar phenomena of the discrete systems. Only the mathematical formalities are different; continuous functions supplant discrete variables and functionals replace the functions of those variables.

### 6.16 Generalization of the Principle of Stationary Potential

The potential of an elastic body is an integral $V$ which depends on a function, the displacement $V$ (or current position $\mathbf{R}$). As such, the integral is a functional. The principle of stationary potential asserts that the functional $V$ is stationary with respect to admissible variations of the function $V$. A modified and useful version of the potential and stationary conditions is credited to H. C. Hu [114] and K. Washizu [115]. The modified potential is also a functional with respect to strains and stresses. Here, we formulate alternative versions incorporating the different strains ($\gamma_{ij}$ and $h_{ij}$) and the associated stresses ($s^{ij}$ and $t^{ij}$).

The potential $V$ consists of the internal energy $U$, and potentials $\Pi_f$ and $\Pi_t$, attributed to conservative external forces. We presume that the latter are explicitly dependent on the position $\mathbf{R}$. However, the internal energy is a functional of the strain: $U = U(\gamma_{ij})$ is an integral of the internal energy $u_0(\gamma_{ij})$ [or $\mathbf{U}(h_{ij})$ and $\mathbf{u}_0(h_{ij})$]. This integral is also implicitly a functional of $\mathbf{R}$ through either one of the following kinematic equations:

$$\gamma_{ij} = \frac{1}{2}(\mathbf{R}_i \cdot \mathbf{R}_j - g_{ij}), \quad (6.122)$$

$$h_{ij} = \frac{1}{2}(\dot{\mathbf{g}}_j \cdot \mathbf{R}_i + \dot{\mathbf{g}}_i \cdot \mathbf{R}_j) - g_{ij}, \quad (6.123)$$

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The latter depends explicitly on the rigid rotation which carries the initial principal lines to their current orientation [see (3.87a, b)]:

\[ \dot{g}_k = r_k^j g_j. \]  

(6.124)

We digress to note that a distinction between the translation \( R \) and rotation \( r \cdot j \cdot k \) is especially meaningful in certain structural problems: (1) The nonlinear behavior of thin bodies (rods, plates, and shells) is often characterized by large changes of rotation, though strains are usually small. (2) The behavior of one small, but finite, element is independent of the translation and rotation, but changes of rotation (a consequence of curvature) must be taken into account in an assembly ([116], [117]).

The modified potential \( V^*(or \bar{V}^*) \) incorporates the internal energy \( U(\gamma_{ij}) \) as a functional of the strain \( \gamma_{ij} \) wherein the strain is subject to the kinematical constraints (6.122) [or (6.123)]. The latter are imposed via Lagrange multipliers \( \lambda_{ij} \) (see, e.g., [118]). Also, the kinematical constraint \( (\mathbf{R} = \mathbf{R}) \) on surface \( s_v \) is enforced by a multiplier \( \tilde{T} \). The modified version of the potential follows:

\[ V^* = \iint_v \left\{ u_0(\gamma_{ij}) - \chi^j \left[ \gamma_{ij} - \frac{1}{2} (\mathbf{R}_i \cdot \mathbf{R}_j - g_{ij}) \right] + \pi_f(\mathbf{R}) \right\} \, dv \]

\[ + \iint_{s_s} \pi_t(\mathbf{R}) \, ds - \iint_{s_v} \tilde{T} \cdot (\mathbf{R} - \bar{\mathbf{R}}) \, ds. \]  

(6.125)

The functional \( V^*(\gamma_{ij}, \chi^j, \tilde{T} \) and \( \mathbf{R} \) is subject to variations of the displacement \( \mathbf{R} \), strains \( \gamma_{ij} \) and multipliers \( \chi^j \) in \( v \), \( \mathbf{R} \) on \( s \) and multipliers \( \tilde{T} \) on \( s_v \). We note that \( V^* = V \) when the kinematical constraints are satisfied; moreover, since \( \gamma_{ij} = \gamma_{ji} \), there is no loss of generality by the restriction \( \chi^j = \chi^i \). The first-order variation follows:

\[ \delta V^* = \iint_v \left\{ \frac{\partial u_0}{\partial \gamma_{ij}} - \chi^j \delta \gamma_{ij} - \delta \chi^j \left[ \gamma_{ij} - \frac{1}{2} (\mathbf{R}_i \cdot \mathbf{R}_j - g_{ij}) \right] \right\} \, dv \]

\[ - \left[ \frac{1}{\sqrt{g}} \left( \sqrt{g} \chi^j \mathbf{G}_j \right)_i \right. - \frac{\partial \pi_t}{\partial X_i} \left. \right] \cdot \delta \mathbf{R} \right\} \, dv \]

\[ - \iint_{s_v} \tilde{T} \cdot [\mathbf{R} - \bar{\mathbf{R}}] \, ds - \iint_{s_v} \left[ \mathbf{T} - \chi^j \mathbf{G}_j n_i \right] \cdot \delta \mathbf{R} \, ds \]

\[ + \iint_{s_s} \left[ \chi^j \mathbf{G}_j n_i + \frac{\partial \pi_t}{\partial X_i} \right] \cdot \delta \mathbf{R} \, ds. \]  

(6.126)
Since the variations are arbitrary \((\gamma_{ij}, \lambda^{ij}, R \text{ in } v, R \text{ on } s, \text{ and } \tilde{T} \text{ on } s_v)\), each bracketed term must vanish; these are the Euler equations:

\begin{align*}
\dot{\lambda}^{ij} &= \frac{\partial u_0}{\partial \gamma_{ij}} \quad \text{in } v, \tag{6.127} \\
\gamma_{ij} &= \frac{1}{2} (R_{ij} \cdot R_{ij} - g_{ij}) \quad \text{in } v, \tag{6.128} \\
\frac{1}{\sqrt{g}} (\sqrt{g} \lambda^{ij} G_{j})_{,i} - \frac{\partial \pi_f}{\partial X^i} t_i &= 0 \quad \text{in } v, \tag{6.129} \\
\lambda^{ij} G_{j} n_i (\equiv T) &= - \frac{\partial \pi_f}{\partial X^i} t_i \quad \text{on } s_v, \tag{6.130} \\
R &= \tilde{R} \quad \text{on } s_v. \tag{6.131}
\end{align*}

Additionally, if \(\delta R \neq 0\) on \(s_v\), then

\[ \tilde{T} = \lambda^{ij} G_{j} n_i \quad \text{on } s_v. \tag{6.132} \]

Equation (6.127) identifies the multiplier \(\dot{\lambda}^{ij} = s^{ij}\). Then (6.129) and (6.130) are recognized as the equilibrium equations (6.111) and (6.112). As anticipated, the variations of the multipliers (\(\dot{\lambda}^{ij} \text{ in } v \text{ and } \dot{\tilde{T}} \text{ on } s_v\)) provide the kinematical constraints (6.128) in \(v\) and (6.131) on \(s_v\). According to (6.132), the multiplier \(\tilde{T}\) emerges as the traction on \(s_v\). Note: The latter (6.132) is not a condition imposed on \(s_v\), since \(\delta R = 0 \ (R = \tilde{R}) \text{ on } s_v\). We have merely admitted such variation to reveal the identity of \(\tilde{T}\).

Notice that the generalized potential \(V^* (\gamma_{ij}, s^{ij}, R)\) is a functional of all the basic functions, each subject to the required admissibility conditions in \(v\) and on \(s\). Then the stationary conditions are the constitutive equations (6.127), all kinematical equations (6.128) and (6.131), and the equilibrium equations (6.129) and (6.130).

Alternative expressions can be derived if the multiplier \(\tilde{T}\) is eliminated from (6.125) using (6.132), or if \(R\) is not variable (\(R = \tilde{R}, \delta R = 0\) on \(s_v\)).

The foregoing functional (6.126) with the corresponding stationary criteria is cited by K. Washizu [115] and B. Fraeijs de Veubeke [119] as the general principle of stationary potential. The procedure is given by R. Courant and D. Hilbert [120].

To devise the modified potential incorporating the engineering strain \(h_{ij}\) and associated stress \(t^{ij}\), we must enforce the kinematical conditions (3.96)
or (3.99) which also entails the rotation of (3.87) (see Section 3.14), specifically, the variation of vector $\dot{g}_i$. The latter is an infinitesimal rigid rotation; accordingly, it can be represented by a vector $\delta \omega$:

$$\delta \dot{g}_i = \delta \omega \times \dot{g}_i = \delta \bar{\omega}^k e_{kin} \dot{g}_i^k \equiv \delta \bar{\omega}_n \dot{g}_n^i.$$  

To implement the variational procedure, we also recall the relation between the rotated vector $\dot{g}_i$ and the deformed version $G_i$ [see (3.95)]:

$$G_i \equiv R_{j,i} = C_{ji} \dot{g}_j = (h_{ji}^i + \delta_{ji}^i) \dot{g}_j.$$ 

(6.134)

The alternative version of the potential takes the form:

$$V^* = \iiint_v \left[ \bar{\alpha}_0 (h_{ij}) - \bar{\lambda}^{ij} (h_{ij} - \dot{g}_j \cdot R_{,i} + g_{ij}) + \pi_f (R) \right] dv$$

$$+ \int_{s_v} \pi_t (R) ds - \int_{s_v} \bar{T} (R - \bar{R}) ds.$$  

(6.135)

This functional is dependent on position $R$ (in $v$ and $s$), strain $h_{ij}$ (in $v$), multipliers $\bar{\lambda}^{ij}$ (in $v$) and $\bar{T}$ (on $s_v$), and also the rotation (in $v$) according to (3.87). Note that the vector $\delta \omega$ effects the variation $\delta r_{ij} = r_{nj}^i e_{kin} \delta \bar{\omega}^k$; see equation (3.105) or (6.124). Notice too that the rotation is most readily referred to the current state, i.e., $\delta \omega = \delta \bar{\omega}^i \dot{g}_i$.

The first-order variation assumes the form:

$$\delta V^* = \iint_v \left[ \frac{\partial \bar{\alpha}_0}{\partial h_{ij}} \delta h_{ij} - \delta \bar{\lambda}^{ij} \left( h_{ij} - \frac{1}{2} (\dot{g}_i \cdot R_{,j} + \dot{g}_j \cdot R_{,i}) + g_{ij} \right) - \frac{1}{\sqrt{g}} \left( \sqrt{g} \bar{\lambda}^{ij} \dot{g}_j^k \right) \frac{\partial \pi_t}{\partial X_{i}^k} \right] \cdot \delta R$$

$$+ \left[ \bar{\lambda}^{ij} (h_{ij} + \delta_{ij}^i) e_{kj}^p \right] \delta \bar{\omega}^k + \delta \bar{\lambda}^{ij} \frac{1}{2} \left( \dot{g}_i \cdot R_{,j} - \dot{g}_j \cdot R_{,i} \right) \right] dv$$

$$- \int_{s_v} \delta \bar{T} \cdot [R - \bar{R}] ds - \int_{s_v} \left[ \bar{T} - \bar{\lambda}^{ij} \dot{g}_j n_i \right] \cdot \delta R ds$$

$$+ \int_{s_v} \left[ \bar{\lambda}^{ij} \dot{g}_j n_i + \frac{\partial \pi_t}{\partial X_{i}^k} \right] \cdot \delta R ds.$$  

(6.136)
This is the counterpart of (6.126), but includes a term resulting from the rotation $\delta \omega$ of vectors $\dot{\mathbf{g}}_i$. Note that the variation $\delta \tilde{\lambda}^{ij}$ has been split into symmetric $[\delta \tilde{\lambda}^{ij}]$ and antisymmetric $[\delta \tilde{\lambda}^{ij}]$ parts. Because the decomposition is such that $h_{ij} = h_{ji}$ and $\dot{\mathbf{g}}_i \cdot \mathbf{R}_j = \dot{\mathbf{g}}_j \cdot \mathbf{R}_i$ (see Subsection 3.14.1), the antisymmetric part makes no contribution to the variation $\delta V^*$ (6.136).

The independence of the variations $(\delta h_{ij}, \delta \tilde{\lambda}^{ij}, \delta \mathbf{R}, \delta \tilde{T}, \delta \omega)$ requires that each bracketed term vanish; these are the Euler equations:

\begin{equation}
\tilde{\lambda}^{ij} = \frac{\partial \tilde{u}_0}{\partial h_{ij}} \quad \text{in } v, \quad (6.137)
\end{equation}

\begin{equation}
h_{ij} = \frac{1}{2}(\dot{\mathbf{g}}_i \cdot \mathbf{R}_j + \dot{\mathbf{g}}_j \cdot \mathbf{R}_i) - g_{ij} \quad \text{in } v, \quad (6.138)
\end{equation}

\begin{equation}
\frac{1}{\sqrt{g}}(\sqrt{g} \tilde{\lambda}^{ij}\dot{\mathbf{g}}_j)_{,i} - \frac{\partial \pi}{\partial X_i} \dot{i}_i = 0 \quad \text{in } v, \quad (6.139)
\end{equation}

\begin{equation}
\tilde{\lambda}^{ij}(h^p_{ij} + \delta^p_{ij}) = \tilde{\lambda}^{ip}(h^j_{ip} + \delta^j_{ip}) \quad \text{in } v, \quad (6.140)
\end{equation}

\begin{equation}
\tilde{\lambda}^{ij}\dot{\mathbf{g}}_j n_i = -\frac{\partial \pi}{\partial X_i} \dot{i}_i \quad \text{on } s_t, \quad (6.141)
\end{equation}

\begin{equation}
\mathbf{R} = \mathbf{R} \quad \text{on } s_v, \quad (6.142)
\end{equation}

\begin{equation}
\tilde{T} = \tilde{\lambda}^{ij}\dot{\mathbf{g}}_j n_i \quad \text{on } s_v. \quad (6.143)
\end{equation}

The equations [(6.137) to (6.143)] differ from their counterparts [equations (6.127) to (6.132)] in two ways:

1. These are expressed in terms of the engineering strain $h_{ij}$ and the associated stress $t^{ij} = \tilde{\lambda}^{ij}$.

2. The system includes the additional equations (6.140) that are the conditions of equilibrium associated with the rigid rotation $\delta \omega$. In his memorable work, B. Fraeijs de Veubeke [121] calls this result [equations (6.140)] “a disguised form of the rotational equilibrium conditions.”

The ultimate value of any theorem is the basis for greater understanding, further development, and useful applications. To those ends the general principle provides a powerful basis for examining and developing means of...
effective approximations by finite elements. Such applications are elucidated in Chapter 11.

6.17 General Functional and Complementary Parts

A primitive functional of stress \( s^i \) and position \( R \) has the form \[122\]:

\[
P = \int\int\int_v (s^i \cdot R, - f \cdot R) \, dv \]

\[
- \int\int T \cdot R \, ds - \int\int (R - \overline{R}) \cdot T \, ds. \tag{6.144}
\]

As before, \( s^i \) may be represented in any of the alternative forms [e.g., (6.105a, b)]; \( f \) denotes the body force, \( R \) the current position, \( T \) the surface traction, and \( \overline{R} \) the imposed position on surface \( s_v \). If the stress satisfies the equilibrium conditions \([\sqrt{g} s^i, + \sqrt{g} f = 0 \text{ in } v, s^n_i = T \text{ on } s_i] \) and the position is admissible [\( R \) and \( \overline{R}, i \) are continuous in \( v \), \( R = \overline{R} \) on \( s_v \)], then \( P = 0 \).

Now, recall the relation between the density of internal energy (internal energy or free energy) and its complement (enthalpy or Gibbs potential) presented in Section 5.11; in the absence of thermal effects

\[
s^{ij} \gamma_{ij} = u_0(\gamma_{ij}) + u_{c0}(s^{ij}), \tag{6.145a}
\]

\[
t^{ij} h_{ij} = \bar{u}_0(h_{ij}) + \bar{u}_{c0}(t^{ij}). \tag{6.145b}
\]

Hereafter, we refer to these as internal density \( (u_0 \text{ or } \bar{u}_0) \) and complementary density \( (u_{c0} \text{ or } \bar{u}_{c0}) \).

With the requisite continuity

\[
s^i \cdot R, = s^{ij}(\gamma_{ij} + \frac{1}{2} g_{ij}) + \frac{1}{2} s^i \cdot R, = t^{ij}(h_{ij} + g_{ij}). \tag{6.146a}
\]

For the elastic body [see (6.145a, b)],

\[
s^i \cdot R, = u_0 + u_{c0} + \frac{1}{2} s^{ij} g_{ij} + \frac{1}{2} s^i \cdot R, = \overline{u}_0 + \bar{u}_{c0} + \bar{t}^i. \tag{6.146b}
\]
In accordance with (4.26c), the components \( t_{ij} \) are associated via the initial metric, i.e., \( t_{ij}^c = g_{jk} t_{ik}^c \).

When (6.146a, b) are substituted into (6.144), one obtains

\[
\mathcal{P} = \iiint_v (u_0 + u_{c0} + \frac{1}{2} s^{ij} g_{ij} + \frac{1}{2} s_i \cdot R,_{i} - f \cdot R) \, dv \\
- \int_s T \cdot R \, ds - \int_{s_v} (R - \overline{R}) \cdot T \, ds,
\]

or

\[
\mathcal{P} = \iiint_v (\bar{u}_0 + \bar{u}_{c0} + t_{ij}^c - f \cdot R) \, dv \\
- \int_s T \cdot R \, ds - \int_{s_v} (R - \overline{R}) \cdot T \, ds.
\]

If the applied loads (\( f \) in \( v \) and \( T \) on \( s_t \)) are “dead” loads (constant), then we recognize a part of each (\( \mathcal{P} \) and \( \mathcal{P} \)) which is the potential in one of the forms:

\[
\mathcal{V} = \iiint_v (u_0 - f \cdot R) \, dv - \int_s T \cdot R \, ds \quad (6.147a)
\]

or

\[
\overline{\mathcal{V}} = \iiint_v (\bar{u}_0 - f \cdot R) \, dv - \int_{s_t} \overline{T} \cdot R \, ds \quad (6.147b)
\]

In the manner of (6.145a, b) we define total complementary potentials such that

\[
\mathcal{P} \equiv \mathcal{V} + \mathcal{V}_c \equiv \overline{\mathcal{V}} + \overline{\mathcal{V}}_c. \quad (6.148a, b)
\]

Then

\[
\mathcal{V}_c \equiv \iiint_v (u_{c0} + \frac{1}{2} s^{ij} g_{ij} + \frac{1}{2} s_i \cdot R,_{i} ) \, dv \\
- \int_{s_v} T \cdot R \, ds - \int_{s_v} T \cdot (R - \overline{R}) \, ds, \quad (6.149a)
\]
\[ \bar{V}_c = \iiint_v \left( \bar{u}_c + s^i \cdot \hat{g}_i \right) \, dv \]
\[ - \int_{s_v} T \cdot R \, ds - \int_{s_v} T \cdot (R - \bar{R}) \, ds. \] (6.149b)

Apart from an irrelevant boundary integral, the form (6.149b) was derived by B. Fraeijs de Veubeke [121]; form (6.149a) was obtained by G. A. Wempner [122].

The reader is forewarned that B. Fraeijs de Veubeke’s, and most earlier works, assign the opposite sign to the complementary potential. Our definition and our sign are chosen for the following reasons: First, we note the analogies [10] between the densities \((u_0 \text{ and } u_c, \bar{u}_0 \text{ and } \bar{u}_c)\) of (6.145a, b) and the total potentials \((V \text{ and } \bar{V}_c, V \text{ and } \bar{V}_c)\) in (6.148a, b). Moreover, in both cases, the left sides of these equations are entirely independent of the material properties. Most important however is the fact that the integral \(P\) vanishes for every admissible equilibrium state. Recall that the potential \((V \text{ or } \bar{V})\) is a relative minimum in a state of stable equilibrium. It follows that the complementary potential is a relative maximum. Of course, this holds only for admissible variations.

### 6.18 Principle of Stationary Complementary Potential

Two general forms of the complementary potential are \(V_c\) and \(\bar{V}_c\) defined by (6.149a and b), respectively. The former is expressed in terms of the Kirchhoff-Trefftz stress \(\{V_c(s^{ij})\}\); the latter in terms of the Jaumann stress \(\{V_c(t^{ij})\}\). Neither is to be viewed as a functional of strain \((\gamma_{ij} \text{ or } h_{ij})\). However, the stress components \((s^{ij} \text{ or } t^{ij})\) are dependent upon a base vector \((G_i \text{ or } \hat{g}_i)\) which is subject to rotation. Stated otherwise, a variation of the vector \(s^i\) is inherently effected by the basis, as well as the stress component. Specifically,

\[ \delta s^i = \delta s^{ij} G_j + s^{ij} \delta \omega \times G_j = \delta t^{ij} \hat{g}_j + t^{ij} \delta \omega \times \hat{g}_j, \]

\[ \delta s^i = \delta s^{ij} G_j + s^{ij} \delta \omega^k E_k j n G^n = \delta t^{ij} \hat{g}_j + t^{ij} \delta \omega^k e_k j n \hat{g}^n. \] (6.150a, b)

Note: The components of the rotation vector \(\delta \omega\) are \(\delta \omega^k \equiv G^k \cdot \delta \omega\) or \(\delta \omega^k \equiv \hat{g}^k \cdot \delta \omega\) (see Subsection 3.14.3).

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We seek the first-order variations of the complementary potentials. Variations of the stress are subject to admissibility, namely equilibrium conditions:

\[(\delta s^i \sqrt{g})_i = 0 \quad \text{in } v, \] (6.151a)

\[\delta s^i n_i = \delta T \quad \text{on } s_v, \] (6.151b)

\[\delta T = 0 \quad \text{on } s_t. \] (6.151c)

In view of the aforestated constraints on \(\delta s^i\) (in \(v\) and on \(s_v\)) and on \(\delta T\) (on \(s_t\)), we take the variations of (6.149a and b); viz.,

\[\delta V_c \equiv \int \int \int_v \left( \partial u_{c,0} \frac{\partial}{\partial s^{ij}} \delta s^{ij} + \frac{1}{2} \delta s^{ij} g_{ij} + \frac{1}{2} \delta s^i \cdot R_{,i} \right) dv \]

\[- \int \int_{s_v} \delta s^i \cdot R n_i ds - \int \int_{s_v} \delta T \cdot (R - \bar{R}) ds, \] (6.152a)

\[\delta V_c' \equiv \int \int \int_v \left( \frac{\partial u_{c,0}}{\partial t^{ij}} \delta t^{ij} + \delta t^{ij} g_{ij} \right) dv \]

\[- \int \int_{s_v} \delta s^i \cdot R n_i ds - \int \int_{s_v} \delta T \cdot (R - \bar{R}) ds. \] (6.152b)

To the third term of the integral in \(v\) of (6.152a), we perform the integration [see equation (2.73)]:

\[\int \int \int_v \left( \delta s^i \cdot R_{,i} - \frac{1}{2} \delta s^i \cdot G_{,i} \right) dv \]

\[= \int \int_{s_v} \delta s^i n_i \cdot R ds - \int \int_v \left[ \frac{1}{\sqrt{g}} (\sqrt{g} \delta s^i)_i \cdot R + \frac{1}{2} \delta s^i \cdot G_{,i} \right] dv. \] (6.153)

When the constraints (6.151a–c) are enforced and (6.150a) is introduced into (6.153), we obtain from (6.152a)

\[\delta V_c = \int \int \int_v \left[ \left( \frac{\partial u_{c,0}}{\partial s^{ij}} + \frac{1}{2} g_{ij} - \frac{1}{2} G_{ij} \right) \delta s^{ij} - \frac{1}{2} s^{ij} E_{kji} \delta \omega^k \right] dv \]

\[- \int \int_{s_v} \delta T \cdot (R - \bar{R}) ds. \] (6.154a)

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By considering $\delta T = \delta s^i n_i = 0$ on $s_v$, the first integral on $s_v$ in (6.152b) assumes the following form:

$$
\iint_{s_v} \delta s^i \cdot R n_i \, ds = \iint_{s_v} \delta s^i \cdot R n_i \, ds = \iint_{v} \left[ \frac{1}{\sqrt{g}} (\sqrt{g} \delta s^i)_i \cdot R + \delta s^i \cdot G_i \right] dv.
$$

Since $\delta s^i$ is expressed by (6.150b) and also satisfies the equilibrium requirement (6.151a), we obtain

$$
- \iint_{s_v} \delta s^i \cdot R n_i \, ds = - \iint_{v} \left( \delta t^{ij} \dot{g}_j \cdot G_i + t^{ij} \delta \omega^k e_{kj} \dot{g}_n \cdot G_i \right) dv.
$$

When this result is substituted into (6.152b) and $G_i$ is expressed by (6.134), we obtain

$$
\delta V_c = \iint_{v} \left[ \left( \frac{\partial u_{c0}}{\partial t} + g_{ij} - \dot{g}_j \cdot G_i \right) \delta t^{ij} - t^{ij} (h_i^n + \delta l^i) \delta \omega^k e_{kj} \right] dv
$$

$$
 \quad - \iint_{s_v} \delta T \cdot (R - R) \, ds.
$$

(6.154b)

Since the variations of stress components and rotations of bases are independent in (6.154a, b), we have the respective stationary (Euler) equations:

$$
\{ \delta V_c = 0 \} \iff \begin{cases} 
\frac{1}{2} (G_{ij} - g_{ij}) = \gamma_{ij} = \frac{\partial u_{c0}}{\partial s^{ij}} & \text{in } v, \\
\delta s^i = \dot{s}^i & \text{in } v, \\
R = \overline{R} & \text{on } s_v.
\end{cases}
$$

(6.155a)
The latter [(6.154b), (6.155b)] were obtained by B. Fraeijs de Veubeke [121]; the former by G. A. Wempner [122]. We note that the complete system of equations is given by the stationary conditions of \( V \) (or \( \nabla V \)) and \( V_c \) (or \( \nabla V_c \)). The stationary conditions for the potential (variation of position \( R \)), enforce equilibrium of stress \( s^i \) in \( v \) and on \( s_t \). The stationary conditions for the complementary potential (variation of stress \( s^i \), both components and basis) enforce the kinematic conditions of (6.155a, b), but also the remaining conditions of equilibrium, viz., those which assert the equilibrium of moment \( (s^{ij} = s^{ji} \text{ or } t^{ij} C^n_i = t^{jn} C^i_j) \).

6.19 Extremal Properties of the Complementary Potentials

To the initial integrands of \( V_c \) and \( \nabla V_c \) of (6.149a, b) we add the terms:

\[
+ \frac{1}{2} s^i \cdot R, i - \frac{1}{2} s^{ij} G_{ij} = 0, \quad (6.156a)
\]

and

\[
+ s^i \cdot R, i - t^{ij} C_{ij} = 0, \quad (6.156b)
\]

respectively. The first term can be integrated by parts such that

\[
\iint_v s^i \cdot R, i dv = \int_s T \cdot R ds - \iiint_v \frac{1}{\sqrt{g}} (\sqrt{g} s^i), i \cdot R dv.
\]

The final term vanishes for an equilibrated state in the absence of body force \( f \). Then, the introduction of (6.156a, b) into (6.149a, b) yields the
alternative forms:

\[
V_c = \iiint_v \left[ u_{c0} - s^{ij} \gamma_{ij} \right] dv + \iint_{s_t} T \cdot R ds, \quad (6.157a)
\]

\[
\overline{V}_c = \iiint_v \left[ \bar{u}_{c0} - t^{ij} \gamma_{ij} \right] dv + \iint_{s_t} T \cdot R ds. \quad (6.157b)
\]

Here, the final terms (integrals on \( s_v \)) have been omitted; those merely assert the forced condition \( R = \overline{R} \) on \( s_v \).

Observe that the first integrals of \( V_c \) and \( \overline{V}_c \) are the negatives of the internal energies, \(-U\) and \(-\overline{U}\). Note too that the stress \( T \) is not subject to variation on \( s_t \). As the forms (6.149a, b), the forms (6.157a, b) are stationary with respect to the equilibrated stresses as the stresses appear explicitly in those functionals.

Recall the previous assessment of functionals \( P \) and \( \overline{P} \), the parts \( V \) and \( \overline{V} \), and parts \( V_c \) and \( \overline{V}_c \). The minimum of potential \( V \) (or \( \overline{V} \)) implies a maximum of the complementary potential \( V_c \) (or \( \overline{V}_c \)). However, those extremals are attained with the admissible variations of all variables. In the absence of body force \( f \), \(-V_c = V\), and \(-\overline{V}_c = \overline{V}\) [see (6.147a, b)]. As previously noted (see Section 6.17), the complementary potential is customarily the negative of \( V_c \) (or \( \overline{V}_c \)). In accord with such custom, let

\[
Q_c \equiv -V_c = \iiint_v u_{0} dv - \iint_{s_t} T \cdot R ds, \quad (6.158a)
\]

\[
\overline{Q}_c \equiv -\overline{V}_c = \iiint_v \bar{u}_{0} dv - \iint_{s_t} T \cdot R ds. \quad (6.158a)
\]

In the equilibrium state, these functionals \( Q_c \) and \( \overline{Q}_c \) are a relative minimum with respect to all admissible variations. It is the latter version (minimum complementary potential) which is most cited in earlier works.

6.20 Functionals and Stationary Theorem of Hellinger-Reissner

The functionals of Hellinger-Reissner [123], [124] are the complements of the modified potential of Hu-Washizu. Much as the potential entails the internal density \((u_0 \text{ or } \bar{u}_0)\), a complementary functional incorporates
the complementary density \((u_{c0} \text{ or } \bar{u}_{c0})\). Two forms may be deduced from (6.149a, b). It is only necessary to apply the Green/Gauss integration to the first integral on surface \(s_v\):

\[
\int_{s_v} T \cdot R \, ds = \int \int_v \left[ \frac{1}{\sqrt{g}} (s^i \sqrt{g})_{,i} \cdot R + s^i \cdot R_{,i} \right] \, dv - \int_{s_t} T \cdot R \, ds.
\]

Here, an admissible stress state satisfies equilibrium; \(T = s_i n_i\) on \(s\) and \((\sqrt{g} s^i)_{,i} = -\sqrt{g} f\) in \(v\). With these requirements, the complementary potentials \((\mathcal{V}_c\text{ and } \bar{\mathcal{V}}_c)\) of (6.149a, b) assume the forms:

\[
\mathcal{V}_c = \int \int_v [u_{c0} - \frac{1}{2} s^{ij}(R_{,j} \cdot R_{,i} - g_{ij}) + f \cdot R] \, dv \\
+ \int_{s_t} T \cdot R \, ds - \int_{s_v} (R - \bar{R}) \cdot T \, ds, \quad (6.159a)
\]

\[
\bar{\mathcal{V}}_c = \int \int_v [\bar{u}_{c0} - t^{ij}(\dot{g}_j \cdot R_{,i} - g_{ij}) + f \cdot R] \, dv \\
+ \int_{s_t} T \cdot R \, ds - \int_{s_v} (R - \bar{R}) \cdot T \, ds. \quad (6.159b)
\]

The functionals \(\mathcal{V}_c^*(s^{ij}, R)\) and \(\bar{\mathcal{V}}_c^*(t^{ij}, R)\), dependent on stress and displacement, are stationary with respect to those functions if, but only if, the displacement-stress relations and equilibrium conditions are satisfied; these are the Euler equations:

\[
\frac{1}{2}(R_{,i} \cdot R_{,j} - g_{ij}) = \frac{\partial u_{c0}}{\partial s^{ij}} \quad \text{in } v, \quad (6.160a)
\]

\[
(\sqrt{g} s^{ij} G_j)_{,i} + \sqrt{g} f = 0 \quad \text{in } v, \quad (6.160b)
\]

\[
s^{ij} G_{ni} = T \quad \text{on } s_t, \quad (6.160c)
\]

\[
R = \bar{R} \quad \text{on } s_v, \quad (6.160d)
\]

or

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In either case, one could accept the symmetry of the stress components ($s^{ij} = s^{ji}$ or $t^{ij} = t^{ji}$). Again, because $g_i \cdot R_{,j} = g_j \cdot R_{,i}$, an antisymmetric part of stress $t^{ij}$ does not contribute to the functional. One could also admit the variation of base vectors $G_i$ or $\dot{g}_i$ which provides the equilibrium conditions presented in equations (6.155a, b), $s^{ij} = s^{ji}$ or $t^{ij} C_i^m = t^{in} C_i^j$.

Note: The modified potentials, forms (6.125) and (6.135), are directly related to their complementary counterparts, (6.159a, b), via (6.145a, b), as follows:

\[ V^* = -V_c^*, \quad V^*_c = -V_c^* \]  

(6.162)

A rudimentary form of the functional $V_c^*$ was suggested by E. Hellinger [123]; the present form and stationarity were enunciated by E. Reissner [124]. The form $V_c^*$ was given by G. A. Wempner [122].

6.21 Functionals and Stationary Criteria for the Continuous Body; Summary

The functionals, potential $V$, complementary potential $V_c$, and modified potentials $V^*$ and $V_c^*$ (or $\bar{V}$, $\bar{V}_c$, $\bar{V}^*$, $\bar{V}_c^*$) are dependent on different functions. Here, the unbarred [or barred (\bar{\ )}] symbols signify the dependence upon the alternative strains and stresses, $\gamma_{ij}$ and $s^{ij}$ (or $h_{ij}$ and $t^{ij}$), respectively.

In summary, the various functionals, variables, and properties follow:

- $V = V(R) = V[u_0(\gamma_{ij})] = \bar{V}[u_0(h_{ij})]$  
  (see Sections 6.15 and 6.17)

Admissible variations satisfy kinematic constraints.  
Stationary conditions enforce equilibrium.  
Minimum potential enforces stable equilibrium.
\[ V_c = V_c(s^{ij}) = \nabla_c \left( t^{ij} \right) = V_c \left[ u_{eo}(s^{ij}) \right] = \nabla_c \left[ \bar{u}_{eo}(t^{ij}) \right] \]

(see Sections 6.17 through 6.19)

Admissible variations satisfy equilibrium.
Stationary conditions enforce the geometric constraints.

\[ V^*_c = V^*_c(R, s^{ij}) = \nabla^*_c \left( R, t^{ij} \right) = V^*_c \left[ R, u_{eo}(s^{ij}) \right] = \nabla^*_c \left[ R, \bar{u}_{eo}(t^{ij}) \right] \]

(see Section 6.20)

Stationarity with respect to displacements and stresses provides the equations of equilibrium and a form of constitutive equations (displacement-stress relations), respectively.

\[ V^* = V^* \left( R, \gamma^{ij}, s^{ij} \right) = \nabla^* \left( R, h^{ij}, t^{ij} \right) = V^* \left[ R, \gamma^{ij}, u_{eo}(s^{ij}) \right] = \nabla^* \left[ R, h^{ij}, \bar{u}_{eo}(t^{ij}) \right] \]

(see Section 6.16)

Stationarity with respect to displacements, stresses, and strains provides all governing equations: equilibrium conditions, kinematic and constitutive relations.

Note: In the functionals \( \nabla_c, \nabla^*_c, \) and \( \nabla^* \), the stress expressed in the form \( s^i = t^{ij} \dot{g}_j \) also admits a variation \( \delta \dot{g}_j = \delta \omega \times \dot{g}_j \) which provides the equilibrium criteria (6.140).

6.22 Generalization of Castigliano’s Theorem on Displacement

As originally presented [125], Castigliano’s theorem provided a means to treat the small displacements of Hookean bodies via a criterion of minimum complementary energy. H. L. Langhaar ([10], pp. 126–130) gives a formulation which applies to nonlinear elasticity, but is also limited to small displacements. In either case, that complementary energy is the integral of the complementary density. As such, that is not to be confused with the complementary energy of Sections 6.8, 6.17, and 6.18. Here, we offer modified forms of complementary energy. These are drawn from the integrals of (6.149a, b) and apply to any elastic body.

As before, we give forms in terms of the densities \( u_{eo}(s^{ij}) \) and \( \bar{u}_{eo}(t^{ij}) \) [see (6.145a, b)]. We define [see (6.149a, b)] an internal complementary energy by either of the following:

\[ C(s^{ij}) \equiv \iint_V \left( u_{eo} + \frac{1}{2} s^{ij} g_{ij} + \frac{1}{2} s^{ij} G_{ij} \right) dv, \quad (6.163a) \]
or

\[ \mathcal{C}(t_{ij}) = \iiint_v (\ddot{u}_{i,0} + t^i_i) \, dv. \]  

(6.163b)

We recall that

\[ \frac{\partial u_{i,0}}{\partial s_{ij}} = \gamma_{ij} = \frac{1}{2} (R_i \cdot R_j - g_{ij}), \]  

(6.164a)

\[ \frac{\partial \ddot{u}_{i,0}}{\partial t^{(ij)}} = h_{ij} = \dot{g}_j \cdot R_i - g_{ij}. \]  

(6.164b)

Therefore, the variations of the functionals \( \mathcal{C}(s_{ij}) \) and \( \vec{\mathcal{C}}(t^{ij}) \) assume the forms:

\[ \delta \mathcal{C} = \iiint_v (\delta s_{ij} G_j \cdot R_i) \, dv, \quad (6.165a) \]

\[ \delta \vec{\mathcal{C}} = \iiint_v (\delta t^{ij} \dot{g}_j \cdot R_i) \, dv. \]  

(6.165b)

Note that \( s_{ij} = s_{ji} \) and \( t^{ij} = t^{(ij)} = t_{ji} \) in the foregoing equations.

By partial integration of each, we obtain

\[ \delta \mathcal{C} = -\iiint_v \frac{1}{\sqrt{g}} (\sqrt{g} \delta s^{ij} G_j)_i \cdot R \, dv + \iint_s \delta s^{ij} n_i G_j \cdot R \, ds, \]  

(6.166a)

\[ \delta \vec{\mathcal{C}} = -\iiint_v \frac{1}{\sqrt{g}} (\sqrt{g} \delta t^{ij} \dot{g}_j)_i \cdot R \, dv + \iint_s \delta t^{ij} n_i \dot{g}_j \cdot R \, ds. \]  

(6.166b)

Variations of stress are to satisfy equilibrium; these are the admissibility conditions:

\[ (\sqrt{g} \delta s^{ij} G_j)_i = (\sqrt{g} \delta t^{ij} \dot{g}_j)_i = 0 \quad \text{in} \quad v, \]  

(6.167a, b)

\[ \delta s^{ij} n_i G_j = \delta t^{ij} n_i \dot{g}_j = \delta T \quad \text{on} \quad s. \]  

(6.168a, b)

Here \( \delta T \) denotes the variation of surface tractions on \( s \). Our results follow:

\[ \delta \mathcal{C} = \iint_s \delta T \cdot R \, ds, \quad \delta \vec{\mathcal{C}} = \iint_s \delta T \cdot R \, ds. \]  

(6.169a, b)
To obtain a general version of Castigliano’s theorem, we follow the logic of H. L. Langhaar ([10], pp. 133–135). We suppose that the body is supported on some portion of the surface \( s \), so that rigid-body displacements of the entire system are impossible; there \( R = r \), the initial position. The body may be subjected to certain concentrated tractions on the surface \( s \). In the accepted spirit of engineering practice, we regard such concentrated force as distributed on a small spot. We are concerned only with the resultant, not the distribution, but understand that local stress and deformation are then beyond the scope of our theory. Let \( s_0 \) denote the very small surface which is subjected to the concentrated traction \( T_0 \) and let \( R_0 \) denote the current position of a particle within \( s_0 \). We now examine (6.169a, b) subject only to a variation of the traction on \( s_0 \). Then,

\[
\delta C = R_0^j \int_s n_i \delta s^{ij} \, ds = R_0^j \delta T^j_0,
\]

\[
\delta C = \bar{R}_0^j \int_s n_i \delta t^{ij} \, ds = \bar{R}_0^j \delta T^j_0.
\]

Note differences in the respective vectors: \( R_0^j \equiv G_j \cdot R^0 \), \( \bar{R}_0^j \equiv \hat{g}_j \cdot R^0 \), etc. The variations of stress (\( s^{ij} \) or \( t^{ij} \)) are a consequence of the variations of load (\( T_0^j \) or \( \bar{T}_0^j \)) and are in equilibrium with that variation [according to (6.167a, b) and (6.168a, b)]. It follows that

\[
\delta C = \frac{\partial C}{\partial T^i_0} \delta T^i_0, \quad \delta \bar{C} = \frac{\partial \bar{C}}{\partial \bar{T}^i_0} \delta \bar{T}^i_0,
\]

or

\[
R^0_i = \frac{\partial C}{\partial T^i_0}, \quad \bar{R}^0_i = \frac{\partial \bar{C}}{\partial \bar{T}^i_0}.
\]

Practically, the result (6.170a, b) is useful if one can readily express the stress in terms of external loading. Engineers are familiar with elementary situations, particularly the so-called “statically determinate” problems, and also the method of a “dummy load.” The forms (6.163a, b) are not limited to Hookean bodies, nor small deformations. However, we note that usage of form \( C \) of (6.163a) is more limited than the form \( \bar{C} \) of (6.163b) since the strain \( \gamma_{ij} = (G_{ij} - g_{ij})/2 \) appears explicitly. This points to an inherent disadvantage in the use of the components \( s^{ij} \), as opposed to \( t^{ij} \). The former components (\( s^{ij} = G^j \cdot s^i \)) are based upon the deformed basis (\( G_j \) and \( G^j \)), whereas the latter (\( t^{ij} = \hat{g}^j \cdot s^i \)) are based on
the rigidly rotated basis ($\dot{g}_j$ and $\dot{g}^i$). Hence, the integrand of $C^*$, like the components $t^{ij}$, is entirely divorced from the strain $h_{ij}$.

Let us now define alternative forms of complementary energy, viz.,

$$C^* \equiv \iiint_{V} \left( u_{c0} + \frac{1}{2} s^{ij} g_{ij} + \frac{1}{2} s^{ij} G_{ij} - s^{ij} G_j \cdot g_i \right) \, dv, \quad (6.171a)$$

$$\overline{C}^* \equiv \iiint_{V} \left( \bar{u}_{c0} + t^i - t^{ij} \dot{g}_j \cdot g_i \right) \, dv. \quad (6.171b)$$

In accordance with (6.164a, b) and (6.165a, b),

$$\delta C^* = \iiint_{V} \left[ \delta s^{ij} G_j \cdot (R - r),i \right] \, dv = \iint_{s} \delta T \cdot (R - r) \, ds,$$

$$\delta \overline{C}^* = \iiint_{V} \left[ \delta t^{ij} \dot{g}_j \cdot (R - r),i \right] \, dv = \iint_{s} \delta T \cdot (R - r) \, ds.$$

By the logic leading to (6.170a, b), we obtain similar results in terms of the displacement ($V = R - r$) at the "point":

$$V_{0}^i = \frac{\partial C^*}{\partial T_{0}^i}, \quad \overline{V}_{0}^i = \frac{\partial \overline{C}^*}{\partial T_{0}^i}. \quad (6.172a, b)$$

Again, we must question the limitations of the result and the practicality. To that end, let us rewrite the integrands of $C^*$ and $\overline{C}^*$ in (6.171a, b):

$$C^* = \iiint_{V} \left[ u_{c0} + \frac{1}{2} s^{ij} (G_j - g_j) \cdot (G_i - g_i) \right] \, dv, \quad (6.173a)$$

$$\overline{C}^* = \iiint_{V} \left[ \bar{u}_{c0} + t^{ij} \dot{g}_j \cdot (\dot{g}_i - g_i) \right] \, dv. \quad (6.173b)$$

To grasp the significance of the final products, we recall the motion and deformation which carries the vector $g_j$ to the vector $\dot{g}_j$, and subsequently to $G_i$. These are the rigid rotation of (3.87a, b) and then the deformation of (3.86a–c) and (3.95), viz.,

$$\dot{g}_i = r^{ij} g_j, \quad G_k = (h^i_k + \delta^i_k) \dot{g}_i.$$
In most practical problems (e.g., machines and structures), the strains are very small \(|e_{ij}| \ll 1\) and rotations are small enough to treat them as vectors \((\boldsymbol{\omega}),\) where \(|\boldsymbol{\omega}| \ll 1\). Then, according to (3.164) and (3.165),
\[
\dot{g}_i = g_i + \Omega_{pi}^p g^p,
G_i = g_i + \Omega_{pi}^p g^p + e_{ip} g^p = g_i + \Omega_{pi}^p g^p + \gamma_{ip} g^p,
\]
where \([\text{see (3.163)}], \) \(\Omega_{ij} \equiv \Omega^k e_{kji}\). The requisite approximations follow:
\[
\frac{1}{2} s^{ij} (G_j - g_j) \cdot (G_i - g_i) \approx \frac{1}{2} s^{ij} \left[ \Omega^k_j \Omega_{ki} + O(\gamma \Omega) \right], \quad (6.174)
\]
\[
t^{ij} (\dot{g}_j - g_j) \approx t^{ij} \Omega_{ji}. \quad (6.175)
\]
Since we now consider small strains \([\text{see also (6.155b)}],\) the relevant approximation is \(t^{ij} = t^{ji}\); hence the right sides of (6.175) vanishes. The approximations of \(C^*\) and \(\overline{C}^*\) (6.173a, b) for small strain and moderate rotation follow:
\[
C^* \approx \iint_v (u_{c0} + \frac{1}{2} s^{ij} \Omega^k_j \Omega_{ki}) \, dv, \quad \overline{C}^* \approx \iint_v \bar{u}_{c0} \, dv. \quad (6.176a, b)
\]
Once again, we detect merit in the decomposition of strain and rotation, the use of engineering strain \(e_{ij}\) and the associated stress \(t^{ij}\), since \(\overline{C}^* = C^*(t^{ij})\) does not depend explicitly on strain or rotation. All of the foregoing complementary energies \((C, \overline{C}, \overline{C}^*, \text{ or } C^*)\) are independent of the elastic properties.

### 6.23 Variational Formulations of Inelasticity

Any process of inelastic deformation is nonconservative. One can form functionals, but not potentials, neither in a physical nor mathematical sense; the Gâteaux variations [126] of such functionals provide, as Euler equations, the equations that govern the inelastic deformation of the body. Such a formulation of an inelastic problem can be useful, if only as a consistent means to obtain a discrete model of the continuous body, e.g., a finite-element assembly.

In Section 5.12, one formulation of inelasticity presumes the existence of a free energy \(\mathcal{F}\) (per unit mass \(\rho_0\)); here we employ the energy density per unit volume \(v\), i.e.,
\[
F(\gamma_{ij}, \gamma_{ij}^N, T) = \rho_0 \mathcal{F}.
\]

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In general, the free energy $F$ is a functional of the strain history; it is path-dependent and is also a function of the inelastic strains $\gamma_{ij}^N$.

Formally, one can write the Gâteaux variation (see, for example, [126]):

$$
\delta W^* \equiv 
\int_{t_0}^t \left\langle \int \int_v \left\{ - s^{ij} \delta \gamma_{ij} + S \delta T + \delta F + s^N_{ij} \delta \gamma_{ij}^N 
- \left[ \gamma_{ij} - \frac{1}{2} (R_{,i}R_{,j} - g_{ij}) \right] \delta s^{ij} 
- \left[ \frac{1}{\sqrt{g}} \left( \sqrt{g} s^{ij} G_j \right)_{,i} + f \right] \cdot \delta R \right\} dv \right\rangle dt
- \int_{t_0}^t \left\{ \int_s (T - n_i s^{ij} G_j) \cdot \delta R \, ds + \int_{s_0} \left( R - \overrightarrow{R} \right) \cdot \delta T \, ds \right\} dt. \quad (6.177)
$$

Equation (6.177) is consistent with the thermodynamic relations of Chapter 5 [see Section 5.12, equations (5.44) to (5.47a, b)]. The variation (6.177) is but an elaboration of the Fréchet variation (6.126) of the potential $V^*$ of (6.125). Here, to accommodate inelastic and time dependent strain, we acknowledge such dependence on time $t$; moreover, in all likelihood, a potential $W^*$ does not exist. The variation vanishes (i.e., $\delta W^* = 0$) if, and only if, the constitutive equations (5.48a–c) of Section 5.12, kinematic equations and dynamic equations are satisfied in $v$ and on $s_v$ and $s_t$ [see equations (6.128) to (6.131) of Section 6.16]. The stationary condition (6.177) is a version of Hamilton’s principle [127]; cf. H. L. Langhaar [10]. If the behavior is time-independent, isothermal and elastic (mechanically conservative), then a potential $W^*$ exists; specifically $W^* = V^*$ of equation (6.125).

Let us turn now to inelasticity as described by the classical concepts of plasticity [see Chapter 5, Sections 5.24 to 5.32]: Yielding is initiated if the stress attains a yield condition, e.g., $\mathcal{Y}(s^{ij}) = \sigma^2$. Inelastic strain ensues, if $\dot{\mathcal{Y}} \equiv (\partial \mathcal{Y}/\partial s^{ij}) \dot{s}^{ij} \geq 0$. Note that the equality sign holds for ideally plastic material. The strain increment $\dot{\gamma}_{ij}$ consists of a recoverable (elastic) part $\dot{\gamma}_{ij}^E$ and a plastic (inelastic) part $\dot{\gamma}_{ij}^P$. The latter follows the normality condition, $\dot{\gamma}_{ij}^P = \dot{\lambda} (\partial \mathcal{Y}/\partial s^{ij})$. Strain hardening alters the parameter $\dot{\lambda}$ and also the yield condition $\mathcal{Y}$.

An incremental version of the Hu-Washizu functional [see (6.125)] has the form:
\[ W^* = \iiint_v \left[ \dot{u}_0 - s^{ij}(\dot{\gamma}_{ij}^E - R_{,i} \cdot \dot{R}_{,j}) - f \cdot \dot{R} - \lambda (\mathcal{Y} - \bar{\sigma}^2) \right] dv \]

\[ - \int_{s_t} T \cdot \dot{R} ds - \int_{s_v} T \cdot (\dot{R} - \bar{R}) ds. \quad (6.178) \]

Here, \( \dot{u}_0 = \dot{u}_0(\gamma_{ij}^E) \) denotes the internal energy; \( \mathcal{Y} = \mathcal{Y}(s^{ij}) = \bar{\sigma}^2 \), the yield condition; and \( W^* = \dot{W}^*(\gamma_{ij}^E, s^{ij}, R, \lambda) \). The usual integration by parts and enforcement of stationarity lead to the requisite governing equations:

\[ s^{ij} = \frac{\partial \dot{u}_0}{\partial \gamma_{ij}^E} \quad \text{in } v, \quad (6.179a) \]

\[ \dot{\gamma}_{ij} \equiv R_{,i} \cdot \dot{R}_{,j} = \dot{\gamma}_{ij}^E + \lambda \frac{\partial \mathcal{Y}}{\partial s^{ij}} \quad \text{in } v, \quad (6.179b) \]

\[ \frac{1}{\sqrt{g}} \left( \sqrt{g} s^{ij} G_{i,j} \right) + f = 0 \quad \text{in } v, \quad (6.179c) \]

\[ \mathcal{Y}(s^{ij}) = \bar{\sigma}^2 \quad \text{in } v, \quad (6.179d) \]

\[ s^{ij} G_{i,n_j} = T \quad \text{on } s_t, \quad (6.179e) \]

\[ \dot{R} = \bar{R} \quad \text{on } s_v. \quad (6.179f) \]

An incremental version of the Hellinger-Reissner functional [see equation (6.159)a] has the form:

\[ \dot{W}_c^* = \iiint_v \left[ \dot{u}_{c0} - s^{ij}(\dot{\gamma}_{ij}^E - R_{,i} \cdot \dot{R}_{,j}) - f \cdot \dot{R} + \lambda (\mathcal{Y} - \bar{\sigma}^2) \right] dv \]

\[ + \int_{s_t} T \cdot \dot{R} ds - \int_{s_v} T \cdot (\dot{R} - \bar{R}) ds. \quad (6.180) \]

Here, \( \dot{W}_c^* = \dot{W}_c^*(s^{ij}, \dot{R}, \lambda); \dot{u}_{c0} = \dot{u}_{c0}(s^{ij}) \) denotes the complementary internal energy; and \( \mathcal{Y} = \mathcal{Y}(s^{ij}) = \bar{\sigma}^2 \), the yield condition. The stationary
conditions follow:

\[
\mathbf{R}_i \cdot \mathbf{\dot{R}}_j (\equiv \mathbf{\dot{\gamma}}_{ij}) = \frac{\partial \mathbf{u}_c}{\partial s^{ij}} + \lambda \frac{\partial \mathcal{Y}}{\partial s^{ij}} \quad \text{in } v, \quad (6.181a)
\]

\[
\frac{1}{\sqrt{g}} \left( \sqrt{g} s^{ij} \mathbf{G}_i \right)_j + \mathbf{f} = \mathbf{0} \quad \text{in } v, \quad (6.181b)
\]

\[
\mathcal{Y}(s^{ij}) = \sigma^2 \quad \text{in } v, \quad (6.181c)
\]

\[
s^{ij} \mathbf{G}_i \mathbf{n}_j = \mathbf{T} \quad \text{on } s_t, \quad (6.181d)
\]

\[
\dot{\mathbf{R}} = \dot{\mathbf{R}} \quad \text{on } s_v. \quad (6.181e)
\]

The latter functional \(\tilde{W}_*\) does not explicitly contain the elastic strain \(\mathbf{\dot{\gamma}}_{ij}^E\).

In a practical application, the variational method provides a means to devise discrete approximations. The formulations and computations must proceed stepwise from the prevailing deformed state. Then, it might prove more expedient to cast the functionals entirely in terms of increments of displacement \(\mathbf{\dot{R}}\), strain \(\mathbf{\dot{\gamma}}_{ij}\), and stress \(s^{ij}\). The latter are linearly related via the tangent modulus \(E^{ijkl}_T\) (see Section 5.32); that relation derives from a quadratic form:

\[
\mathbf{\dot{u}} = \frac{1}{2} E^{ijkl}_T \epsilon_{ij} \mathbf{\dot{\gamma}}_{kl}, \quad \mathbf{\dot{s}}^{ij} = E^{ijkl}_T \mathbf{\dot{\gamma}}_{kl}. \quad (6.182a, b)
\]

These differ from the similar equations of Hookean elasticity: The expression (6.182a) for \(\mathbf{\dot{u}}\) includes dissipation; stress \(s^{ij}\) is nonconservative and \(\mathbf{\dot{\gamma}}_{ij}\) includes the inelastic strain \((\mathbf{\dot{\gamma}}_{ij} = \mathbf{\dot{\gamma}}_{ij}^E + \mathbf{\dot{\gamma}}_{ij}^P)\).

An incremental version of the functional \(\tilde{W}_*\) of (6.178) is a quadratic form in the increments \((\cdot)\). In that form, designated \(\mathbf{\tilde{W}}^*\), we signify quantities of the reference state by a tilde \((\tilde{\cdot})\):

\[
\mathbf{\tilde{W}}^* = \iiint \left\{ \frac{1}{2} E^{ijkl}_T \epsilon_{ij} \mathbf{\dot{\gamma}}_{kl} - s^{ij} \left[ \mathbf{\dot{\gamma}}_{ij} - \frac{1}{2} \left( \mathbf{\dot{\mathbf{R}}}_i \cdot \mathbf{\dot{R}}_j + \mathbf{\dot{R}}_j \cdot \mathbf{\dot{R}}_i \right) \right] 
\right. 
+ \left. \frac{1}{2} \mathbf{s}^{ij} \mathbf{\dot{\mathbf{R}}}_i \cdot \mathbf{\dot{R}}_j - \mathbf{\dot{f}} \cdot \mathbf{\dot{R}} \right\} dv 
- \iint \mathbf{T} \cdot \mathbf{\dot{R}} \mathbf{ds} - \iint \mathbf{\dot{T}} \cdot (\mathbf{\dot{R}} - \mathbf{\mathbf{\dot{R}}} \mathbf{ds}. \quad (6.183)
\]
We require that functional $\hat{W}^*$ be stationary with respect to the functions $\gamma_{ij}^*, \hat{s}^{ij}$, and $\hat{\mathbf{R}}$; the Euler equations follow:

\begin{align}
\hat{s}^{ij} &= E^{ijkl}_{T} \gamma_{kl} \quad \text{in } v, \\
\gamma_{ij}^* &= \frac{1}{2} (\tilde{\mathbf{R}}_i \cdot \hat{\mathbf{R}}_j + \hat{\mathbf{R}}_j \cdot \hat{\mathbf{R}}_i) \quad \text{in } v, \\
\frac{1}{\sqrt{g}} \left[ (\sqrt{g} \hat{s}^{ij} \tilde{\mathbf{R}}_i)_j + (\sqrt{g} \hat{s}^{ij} \hat{\mathbf{R}}_i)_j \right] + \hat{\mathbf{f}} = \mathbf{0} \quad \text{in } v, \\
n_j (\hat{s}^{ij} \tilde{\mathbf{R}}_i + \hat{s}^{ij} \hat{\mathbf{R}}_i) &= \hat{T} \quad \text{on } s_t, \\
\hat{\mathbf{R}} &= \tilde{\mathbf{R}} \quad \text{on } s_v.
\end{align}

The moduli $E^{ijkl}_T$ apply to loading, $\hat{s}^{ij} (\partial Y / \partial s^{ij}) \geq 0$; in the event of unloading, the equations (6.184a–e) apply with $E^{ijkl}_T$ replaced by the elastic moduli. The equations (6.184c, d) are merely the perturbations of (6.179c, e).

An alternative version of (6.183) expresses the strain as the sum of elastic and inelastic parts ($\gamma_{ij}^* = \gamma_{ij}^E + \gamma_{ij}^P$), wherein [see equation (5.131)]

\begin{align}
\hat{s}^{ij} &= E^{ijkl}_{T} \gamma_{ij}^E, \\
\frac{\partial Y}{\partial s^{ij}} \hat{s}^{ij} &= G_P \lambda.
\end{align}

That alternative is a functional of $\gamma_{ij}^E$, $\hat{s}^{ij}$, $\hat{\mathbf{R}}$, and $\lambda$, as follows:

\begin{align}
\hat{W}^{**} &= \iiint_v \left\{ \frac{1}{2} E^{ijkl}_T \gamma_{ij}^E \gamma_{kl}^E + \frac{1}{2} G_P (\lambda)^2 \\
&\quad - \hat{s}^{ij} \left[ \gamma_{ij}^E + \lambda \frac{\partial Y}{\partial s^{ij}} - \frac{1}{2} (\tilde{\mathbf{R}}_i \cdot \hat{\mathbf{R}}_j + \hat{\mathbf{R}}_j \cdot \hat{\mathbf{R}}_i) \right] \\
&\quad + \frac{1}{2} \hat{s}^{ij} \hat{\mathbf{R}}_i \cdot \hat{\mathbf{R}}_j - \hat{\mathbf{f}} \cdot \hat{\mathbf{R}} \right\} dv \\
&\quad - \int_{s_t} \hat{T} \cdot \hat{\mathbf{R}} ds - \int_{s_v} \hat{T} \cdot (\hat{\mathbf{R}} - \tilde{\mathbf{R}}) ds.
\end{align}

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Now, the stationary conditions of $\tilde{W}^*$ are augmented by those derived by the variations of the functions $\gamma_E^{ij}$ and $\lambda$; instead of (6.184a), we obtain (6.185) and (6.186).

A complementary functional is achieved via the Legendre transformation (from $\gamma_E^{ij}$ to $\tilde{s}^{ij}$):

$$\tilde{s}^{ij}\gamma_E^{ij} = \frac{1}{2} E^{ijkl} \gamma_E^{ij} \gamma_E^{kl} + \frac{1}{2} D_{ijkl} \tilde{s}^{ij} \tilde{s}^{kl}. \quad (6.188)$$

The substitution of (6.188) into (6.187) provides an incremental version of the Hellinger-Reissner functional (6.180):

$$\tilde{W}_{c^{**}} = \int_\Omega \left\{ -\frac{1}{2} D_{ijkl} \tilde{s}^{ij} \tilde{s}^{kl} + \frac{1}{2} G_P(\tilde{\lambda})^2 
- \tilde{s}^{ij} \left[ \tilde{\lambda} \frac{\partial Y}{\partial \tilde{s}^{ij}} - \frac{1}{2} \left( \tilde{R}_i \cdot \tilde{R}_j + \tilde{R}_j \cdot \tilde{R}_i \right) \right] 
+ \frac{1}{2} \tilde{s}^{ij} \tilde{R}_i \cdot \tilde{R}_j - \tilde{f} \cdot \tilde{R} \right\} dv 
- \int_{s_t} \tilde{T} \cdot \tilde{R} ds - \int_{s_v} \tilde{T} \cdot (\tilde{R} - \tilde{\mathbf{R}}) ds. \quad (6.189)$$

Strains do not appear explicitly in $\tilde{W}_{c^{**}}$; it is a functional of the incremental stresses $\tilde{s}^{ij}$, displacement $\tilde{\mathbf{R}}$, and $\tilde{\lambda}$. Variation of incremental stress $\tilde{s}^{ij}$ provides a form of constitutive equations, viz.,

$$D_{ijkl} \tilde{s}^{kl} + \tilde{\lambda} \frac{\partial Y}{\partial \tilde{s}^{ij}} = \frac{1}{2} \left( \tilde{R}_i \cdot \tilde{R}_j + \tilde{R}_j \cdot \tilde{R}_i \right). \quad (6.190)$$

The remaining equations, (6.184c–e) and (6.186), hold as before. In the enforcement of the foregoing elastic-plastic relations, it is essential that the yield condition prevails, viz.,

$$\frac{\partial Y}{\partial \tilde{s}^{ij}} \tilde{s}^{ij} = G_P \tilde{\lambda} \geq 0.$$

In the event of unloading [$\tilde{s}^{ij}(\partial Y/\partial \tilde{s}^{ij}) < 0$], the equations apply as well to the elastic response, wherein $\tilde{\lambda} = 0$. 

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