MECHANICS
OF
SOLIDS AND SHELLS
Theories and Approximations

Gerald Wempner
Professor emeritus
Georgia Institute of Technology
Atlanta, Georgia
U.S.A.

Demosthenes Talaslidis
Professor
Aristotle University Thessaloniki
Thessaloniki
Greece
The mechanics of solids and the mechanics of shells have long histories; countless articles and books have been written by eminent scholars. Most books address specific aspects which are necessarily restricted. Many are limited by kinematical assumptions, e.g., small strains and/or small rotations. Others are confined to specific behaviors, e.g., elasticity. This book is also necessarily restricted but in a different way:

This book is intended as a reference for scholars, researchers, and practitioners, to provide a reliable source for the mathematical tools of analysis and approximation. As such, those aspects which are common to all continuous solids are presented in generality. These are the kinematics (Chapter 3), the kinetics (Chapter 4), and the energetics (Chapter 6); the only common limitations are the continuity and cohesion of the medium. The basic quantities, e.g., strains, stresses, energies, and the mathematical relations are developed precisely. To achieve the requisite precision and to reveal the invariant properties of physical entities, the foundations are expressed via the language of vectorial and tensorial analysis (Chapter 2). Additionally, geometrical and physical interpretations are emphasized throughout. Since large rotations play a central role in structural responses, e.g., instabilities, the decomposition of rotations and strains is given special attention and unique geometrical interpretations. Where special restrictive conditions are invoked, they are clearly noted.

With a view toward practical applications, the authors have noted the physical implications of various approximations. In the same spirit, only a few specific results are presented; these are the "exact" solutions of much significance in engineering practice (simple bending and torsion, and examples of actual stress concentrations).

The utility of this book is enhanced by the unification of the various topics: The theories of shells (Chapters 9 and 10) and finite elements (Chapter 11) are couched in the general concepts of the three-dimensional continuous solid. Each of the variational principles and theorems of three dimensions (Chapter 6) has the analogous counterpart in the two dimensions of the shell (Chapter 9).
This book does provide original presentations and interpretations: Principles of work and energy (Chapter 6) include the basic concepts of Koiter’s monumental work on stability at the critical load. Additionally, the various complementary functionals are expressed in terms of the alternative strains and stresses; the various versions are fully correlated and applicable to finite deformations. The presentation of the Kirchhoff-Love shell (Chapter 9) includes original treatments for the plastic behavior of shells.

A final chapter views the finite element as a device for the approximation of the continuous solution. The mathematical and physical attributes are described from that viewpoint. As such, the presentation provides a meaningful bridge between the continuum and the discrete assembly.

Our rationale for the content and the structure of this book is best exhibited by the following sequential decomposition:

- The foundations of all theories of continuous cohesive solids are set forth in generality in the initial Chapters 3 and 4.
- The established theories of elasticity, plasticity, and linear viscoelasticity are presented in the subsequent Chapter 5. These are cast in the context of classical thermodynamics. The specific mathematical descriptions of materials are limited to those which have proven effective in practice and are supported by physical evidence.
- The principles of work and energy are presented in the next Chapter 6; the basic forms are given without kinematical limitations. Only the Castigliano theorem is restricted to small deformations.
- The formulations of linear elasticity and viscoelasticity are contained in Chapter 7. This facet of our subject has evolved to the extent that it is vital to a complete understanding of the mechanics of solids. Here, the basic formulations are couched in the broader context of the general theory. These linear theories are included so that our book provides a more complete source of reference.
- The differential geometry of surfaces is essential to the general theories of shells. Chapter 8 presents the geometric quantities and notations which are employed in the subsequent treatment of shells.
- The mechanics of shells is presented in two parts: Chapter 9 sets forth a theory which is limited only by one kinematic hypothesis; a normal is presumed to remain straight. No restrictions are imposed upon the magnitude of deformations. As such, the mechanics of Chapter 9 encompasses the Kirchhoff-Love theory of the subsequent Chapter 10. The latter includes the traditional formulations of elastic shells, but also presents original theories of elastic-plastic shells.
The final Chapter 11 places the basic notions of finite elements in the context of the mechanics of solids and shells. The most fundamental aspects are described from the mathematical and mechanical perspectives.

We trust that the preceding preview serves to reveal our intended unification of the general foundations, the various theories and approximations.

The authors’ perspectives have been influenced by many experiences, by interactions with colleagues and by the efforts of many predecessors. A few works are most notable: The classic of A. E. H. Love, the insightful works of S. P. Timoshenko, the lucid monograph by V. V. Novozhilov, and the text by A. E. Green and W. Zerna are but a few which have shaped our views. Our grasp of energetic formulations and instability criteria are traceable to the important contributions by E. Trefftz, B. Fraeijs de Veubeke, and W. T. Koiter. Throughout this text, the authors have endeavored to acknowledge the origins of concepts and advances. Inevitably, some are overlooked; others are flawed by historical accounts. The authors apologize to persons who were inadvertently slighted by such mistakes.

On a personal note, Gerald Wempner must acknowledge his most influential teacher: His father, Paul Wempner, was a person with little formal education, but one who demonstrated the value of keen observation and concerted intellectual effort. Demosthenes Talaslidis would like to express his gratitude to his wife Vasso for her tolerance and support throughout preparation of this book. Both authors owe a debt of gratitude to Vasso Talaslidis and Euthalia Papademetriou; their forbearance and hospitality have enabled our collaboration. The authors are also indebted to Professor Walter Wunderlich who encouraged their earlier research. Finally, the authors are obliged to Mrs. Feye Kazantzidou for her careful attention to the illustrations.

Gerald Wempner and Demosthenes Talaslidis
Atlanta and Thessaloniki, July 2002
Contents

1 Introduction
  1.1 Purpose and Scope
  1.2 Mechanical Concepts and Mathematical Representations
  1.3 Index Notation
  1.4 Systems
  1.5 Summation Convention
  1.6 Position of Indices
  1.7 Vector Notation
  1.8 Kronecker Delta
  1.9 Permutation Symbol
  1.10 Symmetrical and Antisymmetrical Systems
  1.11 Abbreviation for Partial Derivatives
  1.12 Terminology
  1.13 Specific Notations

2 Vectors, Tensors, and Curvilinear Coordinates
  2.1 Introduction
  2.2 Curvilinear Coordinates, Base Vectors, and Metric Tensor
  2.3 Products of Base Vectors
  2.4 Components of Vectors
  2.5 Surface and Volume Elements
  2.6 Derivatives of Vectors
  2.7 Tensors and Invariance
  2.8 Associated Tensors
  2.9 Covariant Derivative
  2.10 Transformation from Cartesian to Curvilinear Coordinates
  2.11 Integral Transformations

© 2003 by CRC Press LLC
3 Deformation
3.1 Concept of a Continuous Medium
3.2 Geometry of the Deformed Medium
3.3 Dilation of Volume and Surface
3.4 Vectors and Tensors Associated with the Deformed System
3.5 Nature of Motion in Small Regions
3.6 Strain
3.7 Transformation of Strain Components
3.8 Principal Strains
3.9 Maximum Shear Strain
3.10 Determination of Principal Strains and Principal Directions
3.11 Determination of Extremal Shear Strain
3.12 Engineering Strain Tensor
3.13 Strain Invariants and Volumetric Strain
3.14 Decomposition of Motion into Rotation and Deformation
  3.14.1 Rotation Followed by Deformation
  3.14.2 Deformation Followed by Rotation
  3.14.3 Increments of Rotation
3.15 Physical Components of the Engineering Strain
3.16 Strain-Displacement Relations
3.17 Compatibility of Strain Components
3.18 Rates and Increments of Strain and Rotation
3.19 Eulerian Strain Rate
3.20 Strain Deviator
3.21 Approximation of Small Strain
3.22 Approximations of Small Strain and Moderate Rotation
3.23 Approximations of Small Strain and Small Rotation

4 Stress
4.1 Stress Vector
4.2 Couple Stress
4.3 Actions Upon an Infinitesimal Element
4.4 Equations of Motion
4.5 Tensorial and Invariant Forms of Stress and Internal Work
4.6 Transformation of Stress—Physical Basis
4.7 Properties of a Stressed State
4.8 Hydrostatic Stress
4.9 Stress Deviator
4.10 Alternative Forms of the Equations of Motion
4.11 Significance of Small Strain
4.12 Approximation of Moderate Rotations
4.13 Approximations of Small Strains and Small Rotations; Linear Theory
4.14 Example: Buckling of a Beam

5 Behavior of Materials
5.1 Introduction
5.2 General Considerations
5.3 Thermodynamic Principles
5.4 Excessive Entropy
5.5 Heat Flow
5.6 Entropy, Entropy Flux, and Entropy Production
5.7 Work of Internal Forces (Stresses)
5.8 Alternative Forms of the First and Second Laws
5.9 Saint-Venant’s Principle
5.10 Observations of Simple Tests
5.11 Elasticity
5.12 Inelasticity
5.13 Linearly Elastic (Hookean) Material
5.14 Monotropic Hookean Material
5.15 Orthotropic Hookean Material
5.16 Transversely Isotropic (Hexagonally Symmetric) Hookean Material
5.17 Isotropic Hookean Material
5.18 Heat Conduction
5.19 Heat Conduction in the Hookean Material
5.20 Coefficients of Isotropic Elasticity
5.20.1 Simple Tension
5.20.2 Simple Shear
5.20.3 Uniform Hydrostatic Pressure
5.21 Alternative Forms of the Energy Potentials
5.22 Hookean Behavior in Plane-Stress and Plane-Strain
5.23 Justification of Saint-Venant’s Principle
5.24 Yield Condition
5.25 Yield Condition for Isotropic Materials
5.26 Tresca Yield Condition
5.27 von Mises Yield Criterion
5.28 Plastic Behavior
5.29 Incremental Stress-Strain Relations
5.30 Geometrical Interpretation of the Flow Condition
5.31 Thermodynamic Interpretation
5.32 Tangent Modulus of Elasto-plastic Deformations
5.33 The Equations of Saint-Venant, Lévy, Prandtl, and Reuss
5.34 Hencky Stress-Strain Relations

© 2003 by CRC Press LLC
5.35 Plasticity without a Yield Condition; Endochronic Theory
5.36 An Endochronic Form of Ideal Plasticity
5.37 Viscous Behavior
5.38 Newtonian Fluid
5.39 Linear Viscoelasticity
5.40 Isotropic Linear Viscoelasticity
  5.40.1 Differential Forms of the Stress-Strain Relations
  5.40.2 Integral Forms of the Stress-Strain Relations
  5.40.3 Relations between Compliance and Modulus

6 Principles of Work and Energy
  6.1 Introduction
  6.2 Historical Remarks
  6.3 Terminology
  6.4 Work, Kinetic Energy, and Fourier’s Inequality
  6.5 The Principle of Virtual Work
  6.6 Conservative Forces and Potential Energy
  6.7 Principle of Stationary Potential Energy
  6.8 Complementary Energy
  6.9 Principle of Minimum Potential Energy
  6.10 Structural Stability
  6.11 Stability at the Critical Load
  6.12 Equilibrium States Near the Critical Load
  6.13 Effect of Small Imperfections upon the Buckling Load
  6.14 Principle of Virtual Work Applied to a Continuous Body
  6.15 Principle of Stationary Potential Applied to a Continuous Body
  6.16 Generalization of the Principle of Stationary Potential
  6.17 General Functional and Complementary Parts
  6.18 Principle of Stationary Complementary Potential
  6.19 Extremal Properties of the Complementary Potentials
  6.20 Functionals and Stationary Theorem of Hellinger-Reissner
  6.21 Functionals and Stationary Criteria for the Continuous Body; Summary
  6.22 Generalization of Castigliano’s Theorem on Displacement
  6.23 Variational Formulations of Inelasticity

7 Linear Theories of Isotropic Elasticity and Viscoelasticity
  7.1 Introduction
  7.2 Uses and Limitations of the Linear Theories
  7.3 Kinematic Equations of a Linear Theory
  7.4 Linear Equations of Motion
  7.5 Linear Elasticity

© 2003 by CRC Press LLC
7.6 The Boundary-Value Problems of Linear Elasticity
7.7 Kinematic Formulation
7.8 Solutions via Displacements
7.9 Formulation in Terms of Stresses
7.10 Plane Strain and Plane Stress
7.11 Airy Stress Function
7.12 Stress Concentration at a Circular Hole in a Plate
7.13 General Solution by Complex Variables
7.14 Simple Bending of a Slender Rod
7.15 Torsion of a Cylindrical Bar
   7.15.1 Saint-Venant’s Theory
   7.15.2 Prandtl Stress Function
   7.15.3 Alternative Formulation
   7.15.4 An Example of Stress Concentration
7.16 Linear Viscoelasticity
7.17 Kinematic Formulation
7.18 Quasistatic Problems and Separation of Variables
7.19 Quasistatic Problems in Terms of Displacements
7.20 Quasistatic Problems in Terms of Stresses
7.21 Laplace Transforms and Correspondence with Elastic Problems

8 Differential Geometry of a Surface
8.1 Introduction
8.2 Base Vectors and Metric Tensors of the Surface
8.3 Products of the Base Vectors
8.4 Derivatives of the Base Vectors
8.5 Metric Tensor of the Three-Dimensional Space
8.6 Fundamental Forms
8.7 Curvature and Torsion
8.8 Volume and Area Differentials
8.9 Vectors, Derivatives, and Covariant Derivatives
8.10 Surface Tensors
8.11 Green’s Theorem (Partial Integration) for a Surface
8.12 Equations of Gauss and Codazzi

9 Theory of Shells
9.1 Introduction
9.2 Historical Perspective
9.3 The Essence of Shell Theory
9.4 Scope of the Current Treatment
9.5 Kinematics
9.6 Strains and Stresses
9.7 Equilibrium
9.8 Complementary Potentials
9.9 Physical Interpretations
9.10 Theory of Membranes
9.11 Approximations of Small Strain
9.12 The Meaning of Thin
9.13 Theory of Hookean Shells with Transverse Shear Strain

10 Theories under the Kirchhoff-Love Constraint
10.1 Kinematics
10.2 Stresses and Strains
10.3 Equilibrium
10.4 Compatibility Equations, Stress Functions, and the Static-Geometric Analogy
10.5 Constitutive Equations of the Hookean Shell
10.6 Constitutive Equations of the Thin Hookean Shell
10.7 Intrinsic Kirchhoff-Love Theories
10.8 Plasticity of the Kirchhoff-Love Shell
  10.8.1 Introduction
  10.8.2 Computational Experiments
  10.8.3 Quasi-Shell or Multi-Layer Model
  10.8.4 Approximation by a “Sandwich” Shell
  10.8.5 A Derived Theory
  10.8.6 A Direct Theory
10.9 Strain-Displacement Equations
10.10 Approximation of Small Strains and Moderate Rotations
10.11 Theory of Shallow Shells
10.12 Buckling of Thin Elastic Shells
  10.12.1 Introduction
  10.12.2 Equations of a Critical State
10.13 Refinements-Limitations-References

11 Concepts of Approximation
11.1 Introduction
11.2 Alternative Means of Approximation
11.3 Brief Retrospection
11.4 Concept of Finite Differences
11.5 Stationarity of Functionals; Solutions and Forms of Approximation
11.6 Nodal Approximations via the Stationarity of a Functional
11.7 Higher-Order Approximations with Continuous Derivatives
11.8 Approximation by Finite Elements; Physical and Mathematical Implications

© 2003 by CRC Press LLC
11.9 Approximation via the Potential; Convergence
11.10 Valid Approximations, Excessive Stiffness, and Some Cures
11.11 Approximation via the Modified Potential; Convergence and Efficiency
11.12 Nonconforming Elements; Approximations with Discontinuous Displacements
  11.12.1 Introduction
  11.12.2 Patch Test
  11.12.3 Constraints via Lagrangean Multipliers
  11.12.4 Constraints via Penalty Functions
11.13 Finite Elements of Shells; Basic Features
  11.13.1 Inherent Characteristics
  11.13.2 Some Consequences of Thinness
  11.13.3 Simple Conforming Elements
  11.13.4 Summary
11.14 Supplementary Remarks on Elemental Approximations
11.15 Approximation of Nonlinear Paths

References